

COMPUTING COBORDISM MAPS IN LINK FLOER HOMOLOGY AND THE REDUCED KHOVANOV TQFT

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ABSTRACT. We study the maps induced on link Floer homology by elementary decorated link cobordisms. We compute these for births, deaths, stabilizations, and destabilizations, and show that saddle cobordisms can be computed in terms of maps in a decorated skein exact triangle that extends the oriented skein exact triangle in knot Floer homology. In particular, we completely determine the Alexander and Maslov grading shifts for oriented link cobordisms.

As a corollary, we compute the maps induced by elementary cobordisms between unlinks. We show that these give rise to a $(1+1)$ -dimensional TQFT that coincides with the reduced Khovanov TQFT. Hence, when applied to the cube of resolutions of a marked link diagram, it gives the complex defining the reduced Khovanov homology of the knot. Finally, we define a spectral sequence from (reduced) Khovanov homology using these cobordism maps, and we prove that it is an invariant of the (marked) link.

1. INTRODUCTION

This paper is devoted to developing tools for computing maps induced on link Floer homology by decorated link cobordisms. We also exhibit a connection between these maps and the reduced Khovanov TQFT.

Knot Floer homology, denoted by $\widehat{\mathrm{HFK}}$, and defined independently by Ozsváth-Szabó [OSz04] and Rasmussen [Ras03], is a knot invariant that categorifies the Alexander polynomial. Khovanov [Kho00, Kho03] introduced a categorification of the Jones polynomial, called *Khovanov homology*, and a version of it for marked links, called *reduced Khovanov homology*. Rasmussen [Ras05] conjectured that there exists a spectral sequence from reduced Khovanov homology to knot Floer homology.

Link Floer homology, denoted by $\widehat{\mathrm{HFL}}$, and defined by Ozsváth and Szabó [OSz08], is an extension of $\widehat{\mathrm{HFK}}$ to links that categorifies the multivariable Alexander polynomial. The invariant $\widehat{\mathrm{HFL}}$ is only functorial for decorated links according to Dylan Thurston and the first author [JT12], and moving the decorations around a link often induces a nontrivial automorphism of $\widehat{\mathrm{HFL}}$ by the work of Sarkar [Sar15], see also Zemke [Zem16b]. Hence, when considering link cobordisms, it is necessary to keep track of the decorations. Loosely speaking, the decoration consists of a properly embedded 1-manifold σ in the link cobordism F that divides it into subsurfaces $R_+(\sigma)$ and $R_-(\sigma)$. For the precise definition, see Section 2.2. Decorated link cobordisms

2010 *Mathematics Subject Classification.* 57M27; 57R58.

Key words and phrases. Knot cobordism; Heegaard Floer homology; Khovanov homology; spectral sequence; TQFT.

functorially induce homomorphisms on link Floer homology according to the work of the first author [Juh09].

Via Morse theory, we can write every link cobordism as a composition of elementary link cobordisms. We define these elementary cobordisms in Section 3, and prove that they generate the decorated link cobordism category in Proposition 3.6. In an elementary cobordism (F, σ) , the height function $h: S^3 \times I \rightarrow I$ has at most one generic critical point when restricted to F and to σ , and $\text{Crit}(h|_F) \subseteq \text{Crit}(h|_\sigma)$. In a *birth* cobordism, $h|_F$ has a local minimum. In a *saddle*, $h|_F$ has index one and $h|_\sigma$ has a local maximum. We distinguish *merge* and *split* saddles depending on whether the number of link components decreases or increases by one, respectively. In a *death* cobordism, $h|_F$ has a local maximum. A *stabilisation* is an elementary cobordism where $h|_F$ has no critical points, and $h|_\sigma$ has a local minimum, and a *destabilisation* is when $h|_\sigma$ has a local maximum. A (de)stabilisation is *positive* (resp. *negative*) if the bigon component of $F \setminus \sigma$ lies in $R_+(\sigma)$ (resp. in $R_-(\sigma)$). Finally, an *isotopy* is a cobordism where $h|_F$ and $h|_\sigma$ have no critical points. Our aim is to compute the maps induced by these elementary link cobordisms.

In Section 3, we discuss elementary link cobordisms in full generality. Let $V = \mathbb{F}_2\langle B, T \rangle$ be the 2-dimensional bigraded vector field generated by two homogeneous vectors B (bottom-graded) and T (top-graded), where B lives in Maslov grading $-1/2$ and Alexander grading 0, while T lives in Maslov grading $1/2$ and Alexander grading 0. In Proposition 3.13, we show that if \mathcal{B} is a birth cobordism from (L_0, P_0) to (L_1, P_1) , then there is an isomorphism

$$\widehat{\text{HFL}}(L_1, P_1) \cong \widehat{\text{HFL}}(L_0, P_0) \otimes V$$

such that $F_{\mathcal{B}}(x) = x \otimes T$ for every $x \in \widehat{\text{HFL}}(L_0, P_0)$.

For a death cobordism \mathcal{D} from (L_0, P_0) to (L_1, P_1) , we will show in Proposition 3.14 that there is an isomorphism

$$\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFL}}(L_1, P_1) \otimes V$$

such that $F_{\mathcal{D}}(x \otimes T) = 0$ and $F_{\mathcal{D}}(x \otimes B) = x$ for every $x \in \widehat{\text{HFL}}(L_1, P_1)$.

Now let $W = \mathbb{F}_2\langle b, t \rangle$ be the 2-dimensional bigraded vector field generated by two homogeneous vectors b (bottom-graded) and t (top-graded), where b lives in Maslov grading $-1/2$ and Alexander grading $-1/2$, while t lives in Maslov grading $1/2$ and Alexander grading $1/2$. If \mathcal{X} is a stabilisation from (L, P_0) to (L, P_1) , then we will prove in Proposition 3.7 that there is an isomorphism

$$\widehat{\text{HFL}}(L, P_1) \cong \widehat{\text{HFL}}(L, P_0) \otimes W$$

such that $F_{\mathcal{X}}(x) = x \otimes t$ if \mathcal{X} is positive and $F_{\mathcal{X}}(x) = x \otimes b$ if \mathcal{X} is negative for every $x \in \widehat{\text{HFL}}(L, P_0)$.

We shall see in Proposition 3.8 that if \mathcal{X} is a destabilisation from (L, P_0) to (L, P_1) , then there is an isomorphism

$$\widehat{\text{HFL}}(L, P_0) \cong \widehat{\text{HFL}}(L, P_1) \otimes W$$

such that $F_{\mathcal{X}}(x \otimes t) = 0$ and $F_{\mathcal{X}}(x \otimes b) = x$ if \mathcal{X} is positive, and $F_{\mathcal{X}}(x \otimes t) = x$ and $F_{\mathcal{X}}(x \otimes b) = 0$ if \mathcal{X} is negative.

The maps associated to saddle link cobordisms are more complicated. These maps consist of a contact gluing map and a special cobordism map. We show that the special cobordism map consists of a single 4-dimensional 2-handle attachment. We give a Heegaard diagrammatic description of the induced map, and show that it coincides with the appropriate map in the skein exact triangle of Ozsváth and Szabó [OSz04], or, more precisely, a decorated version of it for decorated links stated in Theorem 2.18. However, the definition of the map still involves the count of holomorphic triangles. More precisely, we prove the following in Theorem 3.11: Suppose that we have a saddle cobordism from (L_0, P_0) to (L_+, P) , and let the link (L, P) be as in Figure 11; i.e., obtained by adding a full right-handed twist to L_+ along the crossing disk. Then, by Theorem 2.18, there is an exact triangle

$$\dots \longrightarrow \widehat{\text{HFL}}(L, P) \xrightarrow{e} \widehat{\text{HFL}}(L_0, P_0) \xrightarrow{f} \widehat{\text{HFL}}(L_+, P) \longrightarrow \dots,$$

and the map f coincides with the cobordism map given by the saddle cobordism from (L_0, P_0) to (L_+, P) . Note that Theorem 2.18 is a decorated, and hence natural extension of the skein exact triangles (7) and (8) of Ozsváth and Szabó [OSz04], and there is no need to distinguish between two cases depending on whether the two strands of L_+ intersecting the crossing disc belong to the same component or not. Furthermore, it also applies to unoriented saddles and to multi-pointed links.

Finally, the map induced by an *isotopy* agrees with a diffeomorphism map according to Subsection 3.1.

In particular, in Theorem 3.17, we compute the Alexander and Maslov grading shifts of any decorated link cobordism $\mathcal{X} = (F, \sigma)$. The map $F_{\mathcal{X}}$ shifts the Maslov grading by $\chi(F)/2 + (\chi(R_+(\sigma)) - \chi(R_-(\sigma)))/2$, and the Alexander grading by $(\chi(R_+(\sigma)) - \chi(R_-(\sigma)))/2$. In particular, the shift in the δ -grading, which is $-\chi(F)/2$, only depends on the topology of F and is independent of the decorations.

Zemke also defined maps induced by decorated link cobordisms in [Zem16a]. The grading shift of his maps, computed in a forthcoming paper [Zem17], seem to agree with what we obtained.

In the special case when we have an elementary cobordism between unlinks, we completely determine the induced maps on link Floer homology. For a summary of our results, see Figure 15.

We give a formula for the distant disjoint union of two decorated link cobordisms. We actually develop a much more general formula that holds for connected sums of sutured manifold cobordisms, cf. Theorem 5.9. Using this, we can compute maps induced by distant disjoint unions of the link cobordisms in Figure 15. By taking compositions and forgetting about the embedding of the link cobordisms into $S^3 \times I$, we obtain a (1+1)-dimensional TQFT that, surprisingly, coincides with the reduced Khovanov TQFT. We explain this connection between link Floer homology and Khovanov homology next.

Khovanov's construction works as follows. Given a link diagram, consider the corresponding *cube of resolutions*: Its vertices are *resolutions* of K , obtained by smoothing all the crossings of the diagram, and its edges correspond to pair-of-pants cobordism between resolutions that differ only at a single crossing.

Let V be a 2-dimensional vector space generated by two vectors v_+ and v_- . (We are using Bar-Natan's notation from [BN02]. Use the identification $v_- \mapsto X$ and $v_+ \mapsto \mathbf{1}$ to recover Khovanov's notation.) Khovanov associates to every resolution of K the vector space $V^{\otimes n}$, where n is the number of components of the resolution, and to every elementary cobordism a linear map. Khovanov's cobordism maps are of two kinds: There is a map for the pair-of-pants cobordism that merges two components, and a map for the pair-of-pants cobordism that splits a component into two. Such cobordism maps are given by a $(1+1)$ -dimensional TQFT, and they turn the vector space obtained by direct summing the vector spaces associated to all resolutions into a chain complex. The homology of this complex is called Khovanov homology.

The reduced version of Khovanov homology is obtained by putting a basepoint on the knot. Consequently, all the resolutions have a marked component. By quotienting each vector space $V^{\otimes n}$ of the complex by $\langle v_- \rangle \otimes V^{\otimes(n-1)}$, where the first factor V is associated to the marked component of the resolution, one obtains a new complex that defines *reduced Khovanov homology*, which is again an invariant of the knot.

We build a $(1+1)$ -dimensional TQFT as follows: We associate to each unlink U_n the \mathbb{F}_2 vector space $\text{HFL}(U_n)$, and to every cobordism the cobordism map in link Floer homology. Our main result linking HFL and Khovanov homology is the following.

Theorem 1.1. *The $(1+1)$ -dimensional TQFT induced by HFL is the same as Khovanov's reduced TQFT. Consequently, the homology of the complex obtained from a cube of resolutions for a knot K by applying the functor HFL is the reduced Khovanov homology of K with \mathbb{F}_2 coefficients.*

To prove Theorem 1.1, we need to overcome the following obstacle. The cobordism maps obtained from the disjoint union formula (i.e.; Theorem 5.9) are expressed in some inconvenient basis of the link Floer homologies. Therefore, we actually need to fix a canonical basis of $\text{HFL}(U_n)$, which is achieved by marking a component of U_n . We then express the cobordism maps that we have computed in this basis, and we find that they are exactly the same as Khovanov's maps induced by the same cobordisms. This concludes the proof of Theorem 1.1.

The last section of the paper is devoted to a side topic: We define a spectral sequence from Khovanov homology and an analogous one from reduced Khovanov homology that are obtained by putting a filtration on the Khovanov and reduced Khovanov complexes, and by endowing them with higher differentials coming from the cobordism maps. We call these filtered complexes the *Khovanov filtered complex* and the *reduced Khovanov filtered complex* associated to a link diagram D . By adjusting Bar-Natan's proof on the invariance of Khovanov homology [BN02], we prove the following.

Theorem 1.2. *The spectral sequence defined by the (reduced) Khovanov filtered complex is an invariant of the (marked) link up to isomorphism.*

This, in particular, implies that all the pages, which are bigraded \mathbb{F}_2 vector spaces $\text{Kh}_{p,q}^r$ (or $\widetilde{\text{Kh}}_{p,q}^r$), are (marked) link invariants. We are aware that Baldwin, Hedden, and Lobb have also studied these spectral sequences. It is an instance of a Khovanov-Floer theory, and hence natural and functorial under link cobordisms.

The limits of the spectral sequences above are unknown. Baldwin and Lobb made some computations to understand the behaviour of the limit, but could not find any non-trivial examples. This led them to conjecture that the spectral sequence always collapses on the second page.

Conjecture 1.3 (Baldwin-Lobb). *The limit $\text{Kh}_{p,q}^\infty$ of the spectral sequence is the ordinary Khovanov homology $\text{Kh}_{p,q} = \text{Kh}_{p,q}^2$.*

Organisation. In Section 2, we recall the definitions of sutured Floer homology and of the cobordism maps associated to decorated link cobordisms, and prove the decorated skein exact triangle. In Section 3, we study the maps induced on $\widehat{\text{HFL}}$ by elementary link cobordisms. In Section 4, we compute the maps induced on $\widehat{\text{HFL}}$ by the pair-of-pants and birth and death cobordisms in Figure 15. In Section 5, we define the disjoint union of two decorated link cobordisms, and we study the behaviour of the cobordism maps under disjoint union. In Theorem 5.9, we prove a formula for a more general case of connected sums of sutured manifold cobordisms. In Section 6, we define a canonical basis for the link Floer homology of a marked unlink in S^3 . The rest of the section is devoted to expressing in this canonical basis the TQFT that arises by applying the functor $\widehat{\text{HFL}}$ to cobordisms between unlinks that are disjoint unions of identity cobordisms and cobordisms from Figure 15. By comparing the $(1+1)$ -dimensional TQFT that we obtain with Khovanov's TQFT, we prove Theorem 1.1. Lastly, Section 7 is devoted to the definition and the proof of invariance of the spectral sequence from Khovanov homology defined using the cobordism maps on $\widehat{\text{HFL}}$. We conclude the section with some remarks about the behaviour of the spectral sequence.

Acknowledgements. We thank John Baldwin, Matt Hedden, Tom Hockenhull, Andrew Lobb, Ciprian Manolescu, Tom Mrowka, Jacob Rasmussen, and Ian Zemke for their comments and suggestions.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 674978). The first author was supported by a Royal Society Research Fellowship. The second author was supported by an EPSRC Doctoral Training Award.

2. PRELIMINARIES

Throughout this paper, \mathbb{F}_2 denotes the field with two elements. We will make extensive use of sutured Floer homology, so we give a short introduction in the following subsection. As it generalises the hat flavour of link Floer homology, it provides us with suitable tools for studying link cobordisms.

2.1. Sutured Floer homology and cobordism maps. Sutured Floer homology [Juh06] is a module over \mathbb{F}_2 associated to a balanced sutured manifold. Thus, we first review what a (balanced) sutured manifold is.

Definition 2.1 ([Gab83, Definition 2.6]). A *sutured manifold* is a compact oriented 3-manifold M with boundary, together with a set $\gamma \subseteq \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains

a homologically non-trivial oriented simple closed curve, called a *suture*. The union of the sutures is denoted by $s(\gamma)$.

Finally, every component of $R(\gamma) = \partial M \setminus \text{Int}(\gamma)$ is oriented in such a way that $\partial R(\gamma)$ is coherent with the sutures. Let $R_+(\gamma)$ (resp. $R_-(\gamma)$) denote the components of $R(\gamma)$ where the normal vector points outwards (resp. inwards).

Definition 2.2 ([Juh06, Definition 2.2]). We say that a sutured manifold (M, γ) is *balanced* if M has no closed components, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, and the map $\pi_0(A(\gamma)) \rightarrow \pi_0(\partial M)$ is surjective.

We will sometimes view γ as a “thickened” oriented 1-manifold, so often we do not distinguish between γ and $s(\gamma)$. It shall be clear from the context which one we mean.

The sutured Floer homology [Juh06] of a balanced sutured manifold (M, γ) is a finite dimensional \mathbb{F}_2 vector space $\text{SFH}(M, \gamma)$. Similarly to Heegaard Floer homology, sutured Floer homology admits a splitting along relative Spin^c structures on M . These are defined in [Juh06, Section 4], and form an affine space $\text{Spin}^c(M, \gamma)$ over $H^2(M, \partial M)$. For each $\mathfrak{s} \in \text{Spin}^c(M, \gamma)$, we have an invariant $\text{SFH}(M, \gamma, \mathfrak{s})$, such that

$$\text{SFH}(M, \gamma) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(M, \gamma)} \text{SFH}(M, \gamma, \mathfrak{s}).$$

In [Juh09], the first author defined a map induced on SFH by a cobordism of balanced sutured manifolds. We briefly recall the definition.

Definition 2.3 ([Juh09, Definition 2.3]). Let (M, γ) be a sutured manifold, and suppose that ξ_0 and ξ_1 are contact structures on M such that ∂M is a convex surface with dividing set γ with respect to both ξ_0 and ξ_1 . Then we say that ξ_0 and ξ_1 are *equivalent* if there is a one-parameter family $\{\xi_t : t \in I\}$ of contact structures such that ∂M is convex with dividing set γ with respect to ξ_t for every $t \in I$. In this case, we write $\xi_0 \sim \xi_1$, and we denote by $[\xi]$ the equivalence class of a contact structure ξ .

Definition 2.4 ([Juh09, Definition 2.4]). Let (M_0, γ_0) and (M_1, γ_1) be sutured manifolds. A *cobordism* from (M_0, γ_0) to (M_1, γ_1) is a triple $\mathcal{W} = (W, Z, [\xi])$, where

- W is a compact oriented 4-manifold with boundary,
- $Z \subseteq \partial W$ is a compact, codimension-0 submanifold with boundary of ∂W such that $\partial W \setminus \text{Int}(Z) = -M_0 \sqcup M_1$,
- ξ is a positive contact structure on Z such that ∂Z is a convex surface with dividing set γ_i on ∂M_i for $i \in \{0, 1\}$.

A cobordism is called *balanced* if both (M_0, γ_0) and (M_1, γ_1) are balanced.

Definition 2.5 ([Juh09, Definition 2.7]). Two cobordisms $\mathcal{W} = (W, Z, [\xi])$ and $\mathcal{W}' = (W', Z', [\xi'])$ from (M_0, γ_0) to (M_1, γ_1) are called *equivalent* if there is an orientation preserving diffeomorphism $\varphi : W \rightarrow W'$ such that $d(Z) = Z'$, $d_*(\xi) = \xi'$, and $d|_{M_0 \cup M_1} = \text{id}$.

Definition 2.6 ([Juh09, Definition 10.4]). A cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (N, γ_1) is called a *boundary cobordism* if

- \mathcal{W} is balanced,
- N is parallel to $M_0 \cup (-Z)$,

- we are given a deformation retraction $r: W \times [0, 1] \rightarrow M_0 \cup (-Z)$ such that $r_0|_W = \text{id}_W$ and $r_1|_N$ is an orientation preserving diffeomorphism from N to $M_0 \cup (-Z)$.

Definition 2.7 ([Juh09, Definition 5.1]). We say that a cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (M_1, γ_1) is *special* if

- \mathcal{W} is balanced,
- $\partial M_0 = \partial M_1$, and $Z = \partial M_0 \times I$ is the trivial cobordism between them,
- ξ is an I -invariant contact structure on Z such that each $\partial M_0 \times \{t\}$ is a convex surface with dividing set $\gamma_0 \times \{t\}$ for every $t \in I$ with respect to the contact vector field $\partial/\partial t$.

In particular, it follows from the last condition that $\gamma_0 = \gamma_1$.

Remark 2.8. According to [Juh09, Definition 10.1], every sutured manifold cobordism can be seen as the composition of a boundary cobordism and a special cobordism. Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1) . Let (N, γ_1) be the sutured manifold $(M_0 \cup (-Z), \gamma_1)$. Then we can think of the cobordism \mathcal{W} as a composition $\mathcal{W}^s \circ \mathcal{W}^b$, where \mathcal{W}^b is a boundary cobordism from (M_0, γ_0) to (N, γ_1) and \mathcal{W}^s is a special cobordism from (N, γ_1) to (M_1, γ_1) .

By [Juh09], every balanced cobordism \mathcal{W} from (M_0, γ_0) to (M_1, γ_1) induces a linear map from $\text{SFH}(M_0, \gamma_0)$ to $\text{SFH}(M_1, \gamma_1)$ that is independent of the equivalence class of \mathcal{W} . As in Remark 2.8, choose a decomposition $\mathcal{W}^s \circ \mathcal{W}^b$ of \mathcal{W} . By [Juh09, Section 5], the special cobordism \mathcal{W}^s naturally yields a map

$$F_{\mathcal{W}^s} : \text{SFH}(N, \gamma_1) \rightarrow \text{SFH}(M_1, \gamma_1).$$

This is constructed via composing maps associated to handle attachments, similarly to the case of cobordisms of closed 3-manifolds [OSz06].

If every component of $N \setminus M_0$ intersects ∂N , then, by [HKM08, Theorem 1.1], there is a contact gluing map

$$\Phi_\xi : \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1),$$

associated to the inclusion of the sutured manifold $(-M_0, -\gamma_0) \subseteq (-N, -\gamma_1)$ and the contact structure ξ on $(-N) \setminus \text{Int}(-M_0)$. The map associated to the cobordism \mathcal{W} is then defined as

$$F_{\mathcal{W}} = F_{\mathcal{W}^s} \circ \Phi_\xi : \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(M_1, \gamma_1).$$

The map Φ_ξ is usually called the *gluing map*, whereas the map $F_{\mathcal{W}^s}$ is usually called the *handle attachment map*, *surgery map*, or *special cobordism map*.

Recall that we supposed that there are no components of $N \setminus M_0$ that do not meet ∂N ; these are called *isolated components*. This requirement was needed for the definition of the gluing map, and also because otherwise N might not be balanced. In the presence of isolated components, the map associated to a cobordism \mathcal{W} is defined as follows: Remove a standard contact ball B^3 (i.e.; a ball with a single suture γ_{B^3} on the boundary) from each isolated component of N , and add them to M_1 . What

we get is a cobordism \mathcal{W}' from (M_0, γ_0) to $(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3})$ with no isolated components. By composing the already defined map

$$F_{\mathcal{W}'} : \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}\left(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3}\right)$$

with the natural isomorphism

$$\text{SFH}\left(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3}\right) \xrightarrow{\sim} \text{SFH}(M_1, \gamma_1),$$

we obtain the cobordism map $F_{\mathcal{W}}$.

The main properties of the cobordism maps are summarised in the following theorem. Here **BSut** denotes the category of balanced sutured manifolds and equivalence classes of cobordisms, whereas **Vect** denotes the category of \mathbb{F}_2 vector spaces.

Theorem 2.9 ([Juh09, Theorem 11.12]). *Sutured Floer homology together with the above cobordism maps define a functor $\text{SFH} : \mathbf{BSut} \rightarrow \mathbf{Vect}$, which is a $(3+1)$ -dimensional TQFT in the sense of Atiyah [Ati88, BT06].*

2.2. Decorated link cobordisms. Sutured Floer homology is particularly well-suited to defining maps induced on link Floer homology by decorated link cobordisms. We recall the necessary definitions, starting with reviewing the real blow-up construction.

Definition 2.10. Suppose that M is a smooth manifold, and let $L \subset M$ be a properly embedded submanifold. For every $p \in L$, let $N_p L = T_p M / T_p L$ be the fiber of the normal bundle of L over p , and let $UN_p L = (N_p L \setminus \{0\}) / \mathbb{R}_+$ be the fiber of the unit normal bundle of L over p . Then the (*spherical*) *blowup* of M along L , denoted by $\text{Bl}_L(M)$, is a manifold with boundary obtained from M by replacing each point $p \in L$ by $UN_p L$. There is a natural projection $\text{Bl}_L(M) \rightarrow M$. For further details, see Arone and Kankaanrinta [AK10].

We now review decorated links.

Definition 2.11 ([Juh09, Definition 4.4]). A *decorated link* is a triple (Y, L, P) , where L is a non-empty link in the connected oriented 3-manifold Y , and $P \subset L$ is a finite set of points. We require that, for every component L_0 of L , the number $|L_0 \cap P|$ is positive and even. Furthermore, we are given a decomposition of L into compact 1-manifolds $R_+(P)$ and $R_-(P)$ such that $R_+(P) \cap R_-(P) = P$.

We can canonically assign a balanced sutured manifold $Y(L, P) = (M, \gamma)$ to every decorated link (Y, L, P) , as follows. Let $M = \text{Bl}_L(Y)$ and $\gamma = \bigcup_{p \in P} UN_p L$. Furthermore,

$$R_{\pm}(\gamma) := \bigcup_{x \in R_{\pm}(P)} UN_x L,$$

oriented as $\pm \partial M$, and we orient γ as $\partial R_+(\gamma)$.

Definition 2.12 ([Juh09, Definition 4.2]). A *surface with divides* (S, σ) is a compact orientable surface S , possibly with boundary, together with a properly embedded 1-manifold σ that divides S into two compact subsurfaces that meet along σ .

Definition 2.13 ([Juh09, Definition 4.1]). For $i \in \{0, 1\}$, let Y_i be a connected, oriented 3-manifold, and let L_i be a non-empty link in Y_i . Then a *link cobordism* from (Y_0, L_0) to (Y_1, L_1) is a pair (X, F) , where

- (1) X is a connected, oriented cobordism from Y_0 to Y_1 ,
- (2) F is a properly embedded, compact, orientable surface in X ,
- (3) $\partial F = L_0 \cup L_1$.

We are now ready to define decorated link cobordisms.

Definition 2.14 ([Juh09, Definition 4.5]). We say that the triple $\mathcal{X} = (X, F, \sigma)$ is a *decorated link cobordism* from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) if

- (1) (X, F) is a link cobordism from (Y_0, L_0) to (Y_1, L_1) ,
- (2) (F, σ) is a surface with divides such that the map

$$\pi_0(\partial\sigma) \rightarrow \pi_0((L_0 \setminus P_0) \cup (L_1 \setminus P_1))$$

is a bijection,

- (3) we can orient each component R of $F \setminus \sigma$ such that whenever $\partial\bar{R}$ crosses a point of P_0 , it goes from $R_+(P_0)$ to $R_-(P_0)$, and whenever it crosses a point of P_1 , it goes from $R_-(P_1)$ to $R_+(P_1)$,
- (4) if F_0 is a closed component of F , then $\sigma \cap F_0 \neq \emptyset$.

Furthermore, we say that two decorated link cobordisms $\mathcal{X} = (X, F, \sigma)$ and $\mathcal{X}' = (X', F', \sigma')$ from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) are *equivalent* if there is an orientation-preserving diffeomorphism $\varphi : X \rightarrow X'$ such that $\varphi(F) = F'$, $\varphi(\sigma) = \sigma'$, $\varphi|_{Y_0} = \text{id}_{Y_0}$ and $\varphi|_{Y_1} = \text{id}_{Y_1}$.

Decorated links and equivalence classes of decorated link cobordisms form a category that is denoted by **DLink**.

According to [Juh09, Definition 4.9], there is a natural sutured manifold cobordism complementary to a decorated link cobordism whose construction we recall in the next definition. For this purpose, we first discuss S^1 -invariant contact structures on circle bundles; see also [Juh09, Section 4]. Let $\pi : M \rightarrow F$ be a principal circle bundle over a compact oriented surface F . An S^1 -invariant contact structure ξ on M determines a dividing set σ on the base F , by requiring that $x \in \sigma$ if and only if ξ is tangent to $\pi^{-1}(x)$, and a splitting of F as $R_+(\sigma) \cup R_-(\sigma)$. The image of any local section of π is a convex surface with dividing set projecting onto σ . In the opposite direction, according to Lutz [Lut77] and Honda [Hon00, Theorem 2.11 and Section 4], given a dividing set σ on F that intersects each component of F non-trivially and divides F into subsurfaces $R_+(\sigma)$ and $R_-(\sigma)$, there is a unique S^1 -invariant contact structure ξ_σ on M , up to isotopy, such that the dividing set associated to ξ_σ is exactly σ , the coorientation of ξ_σ induces the splitting $R_\pm(\sigma)$, and the boundary ∂M is convex.

Definition 2.15 ([Juh09, Definition 4.9]). Let (X, F, σ) be a decorated link cobordism from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) . Then we define the sutured cobordism $\mathcal{W} = \mathcal{W}(X, F, \sigma)$ from $Y_0(L_0, P_0)$ to $Y_1(L_1, P_1)$ as follows. Choose an arbitrary splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$ such that $R_+(\sigma) \cap R_-(\sigma) = \sigma$, and orient F such that $\partial R_+(\sigma)$ (with $R_+(\sigma)$ oriented as a subsurface of F) crosses P_0 from $R_+(P_0)$ to $R_-(P_0)$ and P_1 from $R_-(P_1)$ to $R_+(P_1)$. Then \mathcal{W} is defined to be the triple

$(W, Z, [\xi])$, where $W = \text{Bl}_F(X)$ and $Z = UNF$, oriented as a submanifold of ∂W , finally $\xi = \xi_\sigma$ is an S^1 -invariant contact structure with dividing set σ on F and convex boundary ∂Z with dividing set projecting to $P_0 \cup P_1$.

The contact structure ξ is independent of the splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$. According to [Juh09], if (X, F, σ) and (X', F', σ') are equivalent, so are the cobordisms of sutured manifolds associated to them. Therefore, we obtain a functor

$$\mathcal{W} : \mathbf{DLink} \rightarrow \mathbf{BSut}.$$

Notice that $\mathcal{W}(Y, L, P)$ is what we called $Y(L, P)$. By composing with SFH, we obtain a functor from \mathbf{DLink} to \mathbf{Vect} . This functor is a generalisation of the *link Floer homology* functor $\widehat{\text{HFL}}$ due to the following proposition.

Proposition 2.16 ([Juh06, Proposition 9.2]). *If (Y, L, P) is a decorated link such that P intersects each component of L in exactly 2 points, then*

$$\widehat{\text{HFL}}(Y, L) \cong \text{SFH}(Y(L, P)).$$

The above proposition motivates the following definition.

Definition 2.17. We define the functor

$$\widehat{\text{HFL}} : \mathbf{DLink} \rightarrow \mathbf{Vect}$$

to be the composition $\text{SFH} \circ \mathcal{W}$.

Remark. Given a decorated link cobordism, with a slight abuse of notation, we will also denote the cobordism of sutured manifolds associated to it with the same letter.

Let (F, σ) be a decorated cobordism from L_0 to L_1 . If we choose an orientation of F , then we always orient L_0 as $-\partial F$ and L_1 as ∂F .

2.3. A decorated skein exact triangle. Ozsváth and Szabó [OSz04] defined an oriented skein exact triangle in knot Floer homology. Recall that $\widehat{\text{HFK}}(Y, L)$ of an n -component link L in a closed, connected, oriented 3-manifold Y is obtained as follows: Choose $2n - 2$ points p_1, \dots, p_{n-1} and q_1, \dots, q_{n-1} in L such that if we identify each p_i and q_i in L , then we obtain a connected graph. After attaching 3-dimensional 1-handles to Y with feet at p_i and q_i , and inside each 1-handle taking the connected sum of the link components, we obtain a knot $\kappa(L, \{p_i, q_i\})$ in a 3-manifold $\kappa(Y, \{p_i, q_i\})$ diffeomorphic to $Y \#^{n-1}(S^2 \times S^1)$. Then

$$\widehat{\text{HFK}}(L) := \widehat{\text{HFK}}(\kappa(Y, \{p_i, q_i\}), \kappa(L, \{p_i, q_i\})).$$

The skein exact triangle is stated in terms of this invariant $\widehat{\text{HFK}}$. However, the construction of $\widehat{\text{HFK}}(Y, L)$ is not natural, as it depends on the choice of points p_i and q_i , and is not stated in terms of decorated knots or links. In this subsection, we outline a natural version of the Ozsváth-Szabó skein exact triangle in terms of $\widehat{\text{HFL}}$ that works for multi-pointed and unoriented (but decorated) links.

Let Y be a closed, connected, oriented 3-manifold, and let (L, P) be a decorated link in Y . Suppose that D is a disk embedded in Y that intersects both $R_+(P)$ and $R_-(P)$ transversely in a single point. Let $K = \partial D$, we view this as a knot in the

sutured manifold $(M, \gamma) := Y(L, P)$. Let m , l , and s be oriented simple closed curves on $\partial N(K)$ such that m is a meridian of K , the curve $l = D \cap \partial N(K)$ is a longitude, and s intersects both m and l once algebraically. For $c \in \{m, l, s\}$, let $M_c(K)$ denote the manifold obtained by Dehn filling $M \setminus N(K)$ along c . Then there is a surgery exact triangle

$$\dots \longrightarrow \text{SFH}(M_m(K), \gamma) \xrightarrow{e'} \text{SFH}(M_l(K), \gamma) \xrightarrow{f'} \text{SFH}(M_s(K), \gamma) \longrightarrow \dots$$

Here $M_m(K) = M$, so $\text{SFH}(M_m(K), \gamma) = \widehat{\text{HFL}}(L, P)$. In $M_l(K)$ (see Figure 13), consider the annulus A obtained by capping off $D \setminus N(K)$ with the disk $\{1\} \times D^2 \subset S^1 \times D^2$ whose boundary is glued to l . Then A is a product annulus since $|D \cap R_{\pm}(P)| = 1$. Decomposing $(M_l(K), \gamma)$ along A , we obtain a sutured manifold (M', γ') that coincides with $Y(L_0, P_0)$, where L_0 is obtained from $L \setminus N(D)$ by reconnecting the endpoints parallel to D , and adding one decoration to P along each of these parallel arcs. By the work of Honda, Kazez, and Matić, there is a gluing map

$$\Phi_A: \text{SFH}(Y(L_0, P_0)) \rightarrow \text{SFH}(M_l(K), \gamma),$$

which is an isomorphism as we shall see in Section 3.4.1. By [Juh10, Proposition 5.4], the Spin^c -grading on $\text{SFH}(M_l(K), \gamma)$ is obtained by gluing Spin^c structures along A_+ and A_- , the two sides of A after the decomposition. Finally, $(M_s(K), \gamma) = Y(L_+, P)$, where L_+ is a link in Y obtained by adding a full left-handed twist to L along D . For an illustration showing L , L_0 , and L_+ , see Figure 11. Hence, we obtain the following natural version of the Ozsváth-Szabó skein exact triangle.

Theorem 2.18. *Let Y be a closed, connected, oriented 3-manifold, and let (L, P) be a decorated link in Y . Suppose that D is a disk embedded in Y such that it intersects both $R_+(P)$ and $R_-(P)$ transversely in a single point, and let (L_0, P_0) and (L_+, P) be the decorated links defined above (see Figure 11). Then there is an exact triangle*

$$\dots \longrightarrow \widehat{\text{HFL}}(L, P) \xrightarrow{e} \widehat{\text{HFL}}(L_0, P_0) \xrightarrow{f} \widehat{\text{HFL}}(L_+, P) \longrightarrow \dots,$$

where $e = \Phi_A^{-1} \circ e'$ and $f = f' \circ \Phi_A$.

If $Y = S^3$, we have $|L_0| = |L| \pm 1$, and P consists of two decorations on each component of L and L_+ , then the above exact triangle coincides with that of Ozsváth and Szabó [OSz04], after collapsing the multi-grading onto the main diagonal. The condition $|L_0| = |L| \pm 1$ means that the links L , L_0 , and L_+ can be oriented coherently, which was an assumption made in [OSz04], but which is not necessary for Theorem 2.18. Note that Ozsváth and Szabó distinguish two cases depending on whether the two strands of L_+ meeting D belong to the same component. In both cases, we have $\widehat{\text{HFL}}(L, P) \cong \widehat{\text{HFK}}(L)$ and $\widehat{\text{HFL}}(L_+, P) \cong \widehat{\text{HFK}}(L_+)$. If the two strands belong to the same component of L_+ , then $|L_0| = |L| + 1$, and P_0 consists of exactly two decoration on each component of L_0 , hence $\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFK}}(L_0)$. So, in this case, Theorem 2.18 specializes to exact sequence (7) in [OSz04]:

$$\dots \longrightarrow \widehat{\text{HFK}}(L) \longrightarrow \widehat{\text{HFK}}(L_0) \longrightarrow \widehat{\text{HFK}}(L_+) \longrightarrow \dots$$

On the other hand, if the two strands belong to different components of L_+ , then P_0 consists of six decorations on one component of L_0 and two on all other components,

hence $\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFK}}(L_0) \otimes W \otimes W$. Here $W \otimes W$ is a 4-dimensional vector space with a 1-dimensional summand generated by $b \otimes b$ in Maslov and Alexander gradings -1 , and a 2-dimensional summand generated by $b \otimes t$ and $t \otimes b$ in Maslov and Alexander gradings 0 , and a 1-dimensional summand generated by $t \otimes t$ in Maslov and Alexander gradings 1 . Ozsváth and Szabó [OSz04] denoted this vector space V , though in this paper we use that notation for a different purpose. Hence, in this case, our exact sequence specializes to exact sequence (8) in [OSz04]:

$$\dots \longrightarrow \widehat{\text{HFK}}(L) \longrightarrow \widehat{\text{HFK}}(L_0) \otimes W \otimes W \longrightarrow \widehat{\text{HFK}}(L_+) \longrightarrow \dots$$

In particular, the Alexander and Maslov grading shifts are as in [OSz04], so all the maps preserve the Alexander grading, and e and f decrease the Maslov grading by $1/2$. A completely analogous proof shows that this also holds for arbitrary decorated links in S^3 .

3. ELEMENTARY LINK COBORDISMS

In this section, we compute the maps induced by elementary decorated link cobordisms. Recall that a decorated link is a triple (Y, L, P) , where Y is a manifold, L is a link in Y , and P is a set of points on L ; cf. Definition 2.11. We will usually only write (L, P) when $Y = S^3$. Analogously – see Definition 2.14 – we will denote any decorated link cobordism (X, F, σ) from (L_0, P_0) to (L_1, P_1) , where the ambient 4-manifold $X = S^3 \times I$, by (F, σ) .

Definition 3.1. A decorated cobordism (F, σ) from (L, P_0) to (L, P_1) is a *stabilization* (resp. *destabilization*) if

- $F = L \times I$,
- the only critical point of the height function $h: L \times I \rightarrow I$ on σ is a non-degenerate minimum (resp. maximum),
- all but one component of σ is of the form $\{x\} \times I$ for some $x \in L$.

Now suppose that F is oriented. Then this induces a splitting $F = R_+(\sigma) \cup R_-(\sigma)$ as in Definition 2.15. If the bigon component of $F \setminus \sigma$ lies in $R_+(\sigma)$, then we say that (F, σ) is a positive stabilization (resp. destabilization), and is negative otherwise. Note that this depends on the orientation of F .

Definition 3.2. A *birth* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

- (L_1, P_1) is the split union of (L_0, P_0) and (U_1, P_{U_1}) , the unknot with two decorations,
- $F = F_0 \sqcup D$, where $F_0 = L_0 \times I \subseteq S^3 \times I$ and D is disk such that $\partial D = U_1$ and is embedded in $S^3 \times I$ such that the height function $h|_D$ has a single critical point of index 0,
- if $\sigma_0 = \sigma \cap F_0$ and $\sigma_D = \sigma \cap D$, then σ_0 consists of $|P_0|$ vertical lines, one for each component of $L_0 \setminus P_0$, and σ_D is a single arc on D such that $\text{Crit}(h|_{\sigma_D}) = \text{Crit}(h|_D)$.

Definition 3.3. A *death* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

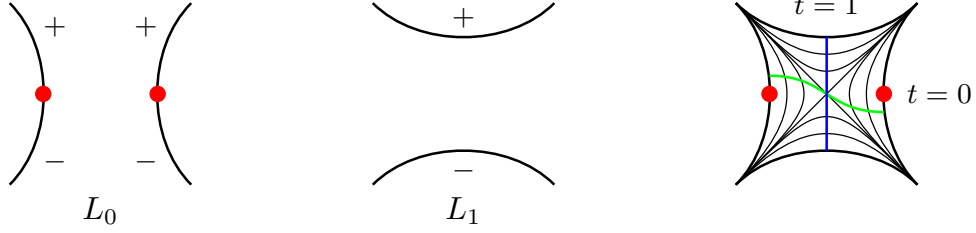


FIGURE 1. This figure illustrates a saddle cobordism. The first two pictures show how L_0 and L_1 differ in a neighbourhood of the square S . The last picture shows how the square S is embedded in $S^3 \times I$, interpolating from L_0 when $t = 0$ to L_1 when $t = 1$. The green arc in the last picture is the decoration σ on S .

- (L_0, P_0) is the split union of (L_1, P_1) and (U_1, P_{U_1}) , the unknot with two decorations,
- $F = F_1 \sqcup D$, where $F_1 = L_1 \times I \subseteq S^3 \times I$ and D is disk such that $\partial D = U_1$ and embedded in $S^3 \times I$ such that the height function $h|_D$ has a single critical point of index 2,
- if $\sigma_1 = \sigma \cap F_1$ and $\sigma_D = \sigma \cap D$, then σ_1 consists of $|P_1|$ vertical lines, one for each component of $L_1 \setminus P_1$, and σ_D is a single arc on D such that $\text{Crit}(h|_{\sigma_D}) = \text{Crit}(h|_D)$.

Notice that, by turning a birth (resp. death) cobordism upside down, one obtains a death (resp. birth) cobordism.

Definition 3.4. A *saddle* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

- $|P_1| = |P_0| - 2$,
- the diffeomorphism $f: S^3 \times \{0\} \rightarrow S^3 \times \{1\}$ defined by $f(x, 0) = (x, 1)$ maps L_0 to L_1 , except in the neighbourhood of a small square $S \approx I \times I$ in S^3 that intersects L_0 and L_1 in the arcs $I \times \partial I$ and $\partial I \times I$, respectively, as illustrated in Figure 1; furthermore, $f^{-1}(P_1) \subseteq P_0$,
- the surface F in $S^3 \times I$ is a product outside a neighbourhood of $S \times I$, and it interpolates between L_0 and L_1 inside $S \times I$, as shown in Figure 1,
- σ consists of an arc supported on the square S shown in green in Figure 1, and outside the square it consists of small parallel translates of $P_0 \times I$, in such a way that

$$\pi_0(\partial\sigma) \rightarrow \pi_0((L_0 \setminus P_0) \sqcup (L_1 \setminus P_1))$$

is a bijection. In particular, $\text{Crit}(h|_\sigma) = \text{Crit}(h|_F)$, and $\text{Crit}(h|_\sigma)$ is a local maximum.

We say that (F, σ) is a *merge* saddle if $|L_1| = |L_0| - 1$, it is a *split* saddle if $|L_1| = |L_0| + 1$, and it is an *unorientable* saddle if $|L_0| = |L_1|$.

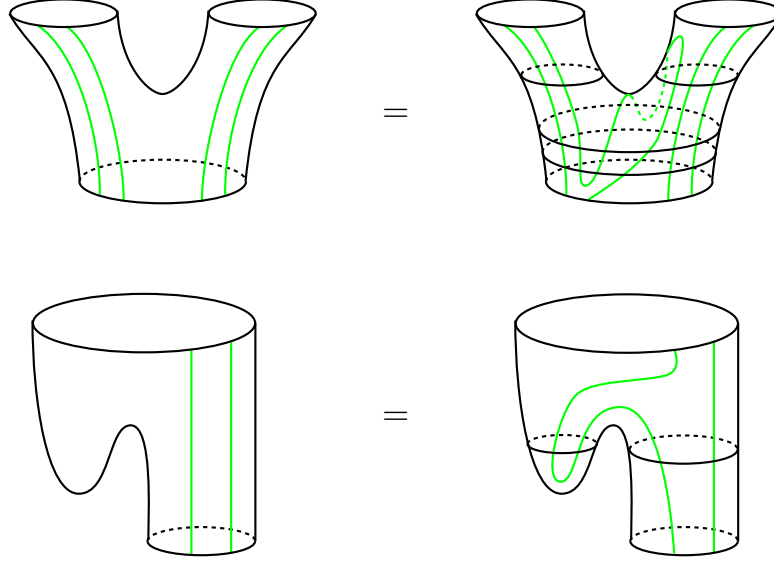


FIGURE 2. Isotoping the dividing set σ such that it passes through the critical points of the height function on F , after which it can be written as a product of elementary cobordisms.

Definition 3.5. Let (F, σ) be a decorated link cobordism from (L_0, P_0) to (L_1, P_1) , and let $h: S^3 \times I \rightarrow I$ be the height function. Then we say that (F, σ) is an *isotopy* if $h|_F$ and $h|_\sigma$ have no critical points.

Proposition 3.6. *The category **DLink** is generated by stabilizations, destabilizations, births, deaths, saddles, and isotopies.*

Proof. By a slight perturbation of F , we can assume $h|_F$ is a generic Morse functions. Then isotope σ such that it passes through the critical points of $h|_F$, and such that $h|_\sigma$ is also a generic Morse function with $\text{Crit}(h|_F) \subset \text{Crit}(h|_\sigma)$. We also require that if p is a saddle point of $h|_F$, then $h|_\sigma$ has a local maximum. Finally, if $t \in I$ is a regular value of $h|_\sigma$, then each component of $h^{-1}(t) \cap F$ contains at least two points of $h^{-1}(t) \cap \sigma$. For an illustration, see Figure 2. The cobordism that we obtain is equivalent to the original. Then let $c_1 < c_2 < \dots < c_n$ be an enumeration of the critical values of $h|_\sigma$, where $n = |\text{Crit}(h|_\sigma)|$. We write $c_0 = 0$ and $c_{n+1} = 1$, and let

$$\varepsilon = \min\{(c_i - c_{i-1})/3 : 1 \leq i \leq n+1\}.$$

We set $t_{2i} = c_i - \varepsilon$ and $t_{2i+1} = c_i + \varepsilon$ for $i \in \{1, \dots, n\}$, and write $t_1 = 0$ and $t_{2n+2} = 1$.

Fix an orientation of F and a splitting $F = R_+(\sigma) \cup R_-(\sigma)$ as in Definition 2.15. Let $L_j = F \cap h^{-1}(t_j)$, and let P_j be a decoration of L_j such that its points alternate with $\sigma \cap L_j$. Then (F, σ) is the product of the cobordisms

$$(F_j, \sigma_j) = (F \cap h^{-1}[t_j, t_{j+1}], \sigma \cap h^{-1}[t_j, t_{j+1}])$$

for $j \in \{1, \dots, 2n+1\}$ from (L_j, P_j) to (L_{j+1}, P_{j+1}) . The splitting $L_j = R_+(P_j) \cup R_-(P_j)$ is chosen such that $\partial R_+(\sigma_j)$ goes from $R_-(P_j)$ to $R_+(P_j)$ as it passes P_j , where $R_+(\sigma_j) = R_+(\sigma) \cap F_j$.

When j is even, $h|_{\sigma_j}$ has a single critical point p_j (that is possibly also a critical point of $h|_{F_j}$), and we can isotope F and σ such that F_j and σ_j are products outside a small neighbourhood of the critical point, and all of F_j is a product if $p \notin \text{Crit}(h|_F)$. Then every (F_j, σ_j) is an elementary cobordism. In particular, it is an isotopy if and only if j is odd. \square

3.1. Link cobordism maps induced by isotopies. Let $\mathcal{X} = (F, \sigma)$ be an isotopy from (L_0, P_0) to (L_1, P_1) , and let $\sigma' \subset F$ be a push-off of σ in a normal direction such that $\partial\sigma' = P_0 \cup P_1$. If we set $L_t = F \cap h^{-1}(t)$ and $P_t = \sigma' \cap h^{-1}(t)$ for $t \in I$, then there is an ambient isotopy $d_t: S^3 \rightarrow S^3$ for $t \in I$ such that $L_t = d_t(L_0)$ and $P_t = d_t(P_0)$. Then d_1 is well-defined up to isotopy, and it follows from [Juh14, Lemma 2.22] that the link cobordism map

$$F_{\mathcal{X}} = (d_1)_*: \widehat{\text{HFL}}(L_0, P_0) \rightarrow \widehat{\text{HFL}}(L_1, P_1)$$

defined in [JT12]. In particular, it preserves both the homological and the Alexander gradings.

3.2. The maps induced by stabilizations and destabilizations.

Proposition 3.7. *Let $\mathcal{S} = (F, \sigma)$ be a stabilization from (L, P_0) to (L, P_1) . We denote by W the bigraded vector space $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where the first summand is generated by the element b in Maslov grading 0 and Alexander grading $-1/2$, and the second summand is generated by the element t in Maslov grading 0 and Alexander grading $1/2$. Then, given an orientation of F , there is a canonical isomorphism*

$$I_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \otimes W \xrightarrow{\sim} \widehat{\text{HFL}}(L, P_1)$$

such that the link cobordism map

$$F_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_1)$$

factorizes as $I_{\mathcal{S}} \circ s_{\mathcal{S}}$, where

$$s_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_0) \otimes W$$

is given by $s_{\mathcal{S}}(x) = x \otimes t$ in case of a positive stabilization, and $s_{\mathcal{S}}(x) = x \otimes b$ in case of a negative stabilization.

Proof. Let $\mathcal{W} = \mathcal{W}(F, \sigma) = (W, Z, \xi)$ be the sutured manifold cobordism complementary to (F, σ) from the sutured manifold $(M, \gamma_0) = \mathcal{W}(L, P_0)$ complementary to (L, P_0) to the sutured manifold $(M, \gamma_1) = \mathcal{W}(L, P_1)$ complementary to (L, P_1) . Here $Z \cong -F \times S^1$ is oriented as the boundary of W , and hence the orientation of F induces an orientation of the S^1 factor. This is the orientation for which S^1 has linking number one with F .

By definition, the cobordism map $F_{\mathcal{W}}$ is the composition of the gluing map

$$\Phi_{-\xi}: \text{SFH}(M, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1),$$

where $N = M \cup (-Z)$, and a special cobordism map

$$F_{\mathcal{W}^s}: \text{SFH}(N, \gamma_1) \rightarrow \text{SFH}(M, \gamma_1).$$

As \mathcal{W}^s is a product, it induces the identity map. So, it suffices to compute $\Phi_{-\xi}$.

We denote by σ_0 the component of σ that contains the critical point of the height function $L \times I \rightarrow I$. Let a be a properly embedded arc in $F \setminus \sigma$ that is parallel to σ_0 , is disjoint from the bigon component of $F \setminus \sigma$, and such that there are two points of P_1 in the rectangular region between a and σ_0 . Then $A = a \times S^1$ is a product annulus in (N, γ_1) . If we decompose (N, γ_1) along A , we obtain the disjoint union of (M, γ_0) and the sutured manifold $(D^2 \times S^1, \gamma)$, where γ consists of four longitudinal sutures. By [Juh10, Proposition 9.1], we have $\text{SFH}(D^2 \times S^1, \gamma) \cong V$, and the isomorphism is uniquely given by the orientation of the S^1 factor, which in turn gives an orientation of $H_1(D^2 \times S^1)$, and hence allows one to distinguish between the two Spin^c -structures in which the two \mathbb{Z}_2 summands live. Hence, as in [Juh10, Proposition 9.2], we obtain a canonical isomorphism

$$(1) \quad \widehat{\text{HFL}}(L, P_0) \otimes W \cong \widehat{\text{HFL}}(L, P_1).$$

We write the gluing map $\Phi_{-\xi}$ as a composition of two gluing maps, which is possible according to [HKM08, Proposition 6.2]. The first gluing map s_S corresponds to the sutured submanifold $(-M, -\gamma_0)$ of $(-M, -\gamma_0) \sqcup (-D^2 \times S^1, -\gamma)$ with the contact structure $-\xi|_{D^2 \times S^1}$ on the difference. The second gluing map I_S corresponds to the sutured submanifold $(-M, -\gamma_0) \sqcup (-D^2 \times S^1, -\gamma)$ of $(-N, -\gamma_1)$, using the contact structure $-\xi$ on $(-M) \setminus ((-N) \sqcup (-D^2 \times S^1))$. By [HKM08, Proposition 6.4], the gluing map

$$I_S: \widehat{\text{HFL}}(L, P_0) \otimes \text{SFH}(D^2 \times S^1, \gamma) \xrightarrow{\sim} \widehat{\text{HFL}}(L, P_1)$$

is an isomorphism.

By construction, the gluing map s_S maps $x \in \widehat{\text{HFL}}(L, P_0)$ to

$$x \otimes \text{EH}(-\xi|_{D^2 \times S^1}) \in \widehat{\text{HFL}}(L, P_0) \otimes \text{SFH}(D^2 \times S^1, \gamma).$$

Recall that $-\xi$ is a positive contact structure on $(-D^2 \times S^1, -\gamma)$. In case of a positive stabilization, the dividing set of $-\xi$ on $-D^2 \times \{\text{pt}\}$ is given by the left-hand side of [HKM08, Figure 14], while it is the right-hand side of [HKM08, Figure 14] in case of a negative stabilization. As explained there, the grading of the contact element of $\text{EH}(-\xi|_{D^2 \times S^1})$ is given by $\chi(R_+) - \chi(R_-)$ for the dividing set on $D^2 \times \{\text{pt}\}$. Hence $\text{EH}(-\xi|_{D^2 \times S^1}) = t$ in case of a positive stabilization and $\text{EH}(-\xi|_{D^2 \times S^1}) = b$ in case of a negative stabilization. \square

Proposition 3.8. *Let $\mathcal{D} = (F, \sigma)$ be a destabilization from (L, P_0) to (L, P_1) . Let W be as in Proposition 3.7. Then, given an orientation of F , there is a canonical isomorphism*

$$I_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_0) \xrightarrow{\sim} \widehat{\text{HFL}}(L, P_1) \otimes W$$

such that the link cobordism map

$$F_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_1)$$

factorizes as $d_{\mathcal{D}} \circ I_{\mathcal{D}}$, where

$$d_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_1) \otimes W \rightarrow \widehat{\text{HFL}}(L, P_1)$$

is given by $d_{\mathcal{D}}(x \otimes b) = x$ and $d_{\mathcal{D}}(x \otimes t) = 0$ in case of a positive destabilization, and by $d_{\mathcal{D}}(x \otimes t) = x$ and $d_{\mathcal{D}}(x \otimes b) = 0$ in case of a negative destabilization.

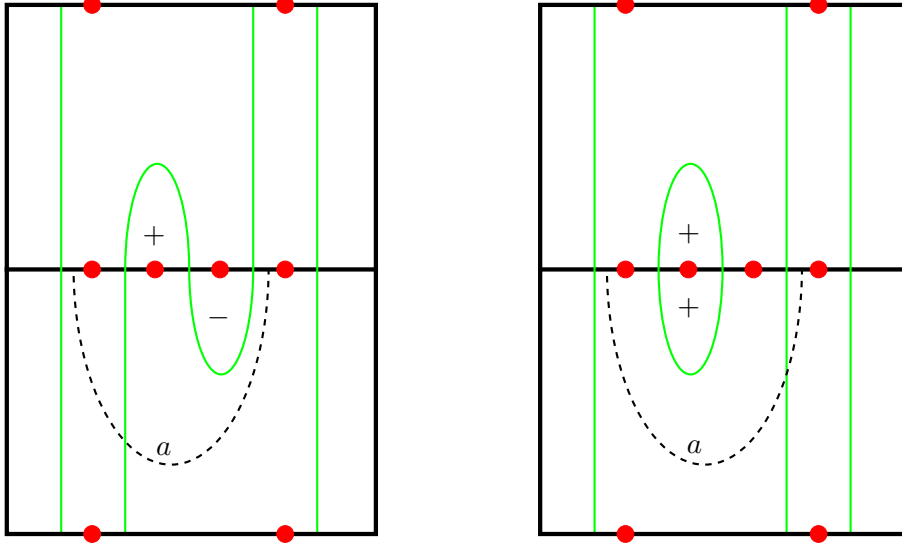


FIGURE 3. The composition of a negative stabilization and positive destabilization on the left, and the composition of a positive stabilization and a positive destabilization on the right.

Proof. Given a positive destabilization $\mathcal{D} = (F, \sigma)$, let $\mathcal{S}_- = (F, \sigma_-)$ be a negative stabilization from (L, P_1) to (L, P_0) such that $\mathcal{D} \circ \mathcal{S}_-$ exists and is as on the left-hand side of Figure 3. Then $\mathcal{D} \circ \mathcal{S}_-$ is the identity cobordism from (L, P_1) to itself, and hence it induces the identity of $\widehat{\text{HFL}}(L, P_1)$.

Now let $\mathcal{S}_+ = (F, \sigma_+)$ be the positive stabilization shown on the right-hand side of Figure 3. Then $F_{\mathcal{D}} \circ F_{\mathcal{S}_+} = 0$, since the dividing set $\sigma \cup \sigma_+$ contains a closed component that bounds a disk, so the corresponding S^1 -invariant contact structure is overtwisted, and hence the gluing map vanishes by Honda et al. [HKM08].

Let a be the properly embedded dashed arc in F shown in Figure 3. The bigon component D of $F \setminus a$ contains the bigon components of $F \setminus \sigma_+$ and $F \setminus \sigma_-$ and the three points of P_0 involved in the destabilization, and intersects both σ_+ and σ_- in two arcs. Then $A = a \times S^1$ is a product annulus in $(M, \gamma_0) = \mathcal{W}(L, P_0)$.

By Proposition 3.7,

$$F_{\mathcal{S}_+} = I_{\mathcal{S}_+} \circ s_{\mathcal{S}_+},$$

where we use the above product annulus A to define the gluing map

$$I_{\mathcal{S}_+} : \widehat{\text{HFL}}(L, P_1) \otimes W \rightarrow \widehat{\text{HFL}}(L, P_1).$$

Let

$$d_{\mathcal{D}} = F_{\mathcal{D}} \circ I_{\mathcal{S}_+}.$$

If we set $I_{\mathcal{D}} := I_{\mathcal{S}_+}^{-1}$, then

$$F_{\mathcal{D}} = d_{\mathcal{D}} \circ I_{\mathcal{D}}.$$

Given $x \in \widehat{\text{HFL}}(L, P_1)$, we obtain that

$$0 = F_{\mathcal{D}} \circ F_{\mathcal{S}_+}(x) = d_{\mathcal{D}} \circ s_{\mathcal{S}_+}(x) = d_{\mathcal{D}}(x \otimes t),$$

as claimed.

In case of \mathcal{S}_- , we repeat the proof of Proposition 3.7 with the same product annulus A to obtain that

$$F_{\mathcal{S}_-} = I_{\mathcal{S}_+} \circ s_{\mathcal{S}_-}.$$

Indeed, the dividing set of the S^1 -invariant contact structure corresponding to σ_- on the bigon D consists of two arcs such that $\chi(R_+) - \chi(R_-) = -1$. Given $x \in \widehat{\text{HFL}}(L, P_1)$, we obtain that

$$x = F_{\mathcal{D}} \circ F_{\mathcal{S}_-}(x) = d_{\mathcal{D}} \circ s_{\mathcal{S}_-}(x) = d_{\mathcal{D}}(x \otimes b),$$

which completes the proof. \square

3.3. The Heegaard diagram arising from a link projection. Ozsváth and Szabó [OSz04] described how to construct a Heegaard diagram from a knot projection. We extend their construction to decorated links; see Figure 4 for an illustration.

Let (L, P) be an oriented decorated n -component link with $|P| = 2\ell$, and let D be a connected projection of L onto $\mathbb{R}^2 \subseteq \mathbb{R}^3$. Let Σ be the boundary of a regular neighbourhood of D in \mathbb{R}^3 . Then Σ is a surface of some genus g , oriented as the boundary of its exterior. Place a basepoint on Σ right above each point of the decoration P . The basepoints are alternately labeled by w_i and z_i such that each component of $R_-(P)$ is oriented from w_i to z_i . We let $\mathbf{w} = \{w_1, \dots, w_\ell\}$ and $\mathbf{z} = \{z_1, \dots, z_\ell\}$.

For each crossing of D , add a β -curve on Σ according to the orientation of the crossing. We call these curves $\beta_1, \dots, \beta_{g-1}$. Furthermore, for each component of $R_-(P)$, add a meridional β -curve that we call $\beta_g, \dots, \beta_{g+\ell-1}$. What we finally get is a set

$$\boldsymbol{\beta} = \{\beta_1, \dots, \beta_{g+\ell-1}\}$$

of attaching circles.

As for the α -curves, draw a curve around each bounded region of $\mathbb{R}^2 \setminus D$, and call these curves $\alpha_1, \dots, \alpha_g$. Furthermore, for each component of $R_-(P)$ except one, add an inessential α -curve that contains z_i and w_i as in Figure 4. We call these “ladybugs,” and are labeled $\alpha_{g+1}, \dots, \alpha_{g+\ell-1}$. We set

$$\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{g+\ell-1}\}.$$

The diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a 2ℓ -pointed Heegaard diagram representing the link L . A sutured Heegaard diagram for the complement of a link is obtained from a 2ℓ -pointed Heegaard diagram by removing a small disc about each of the basepoints w_i and z_i . For this reason, we will use 2ℓ -pointed Heegaard diagrams and sutured Heegaard diagrams for decorated link complements without distinction.

3.4. The map associated to a saddle cobordism. Consider a saddle cobordism (F, σ) from (L_0, P_0) to (L_1, P_1) . Let $(M_0, \gamma_0) = \mathcal{W}(L_0, P_0)$ and $(M_1, \gamma_1) = \mathcal{W}(L_1, P_1)$ denote the sutured manifolds complementary to the decorated links.

We decompose the sutured manifold cobordism complementary to the saddle cobordism into two cobordisms, a *boundary cobordism* \mathcal{W}^b and a *special cobordism* \mathcal{W}^s . Let (N, γ_1) denote the outgoing end of the boundary cobordism. Since F is oriented,

$$N \cong M_0 \cup_{(L_0 \cap S) \times S^1} (S \times S^1),$$

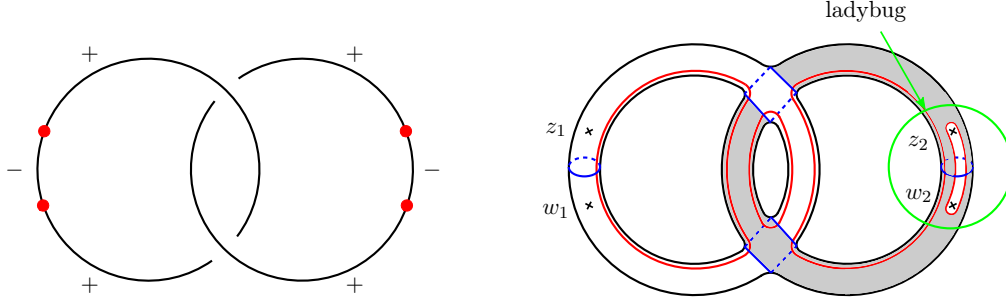


FIGURE 4. The picture on the left shows a projection of the Hopf link with two decorations on each component. From this projection, one can construct the Heegaard diagram on the right-hand side. The grey region on the right-hand side is the periodic domain corresponding to the second link component.

where S is the square shown in Figure 1.

3.4.1. *The boundary cobordism associated to a saddle.* The boundary cobordism associated to a saddle is relatively easy to describe. Consider the annulus $A = b \times S^1 \subseteq N$, where $b = I \times \{1/2\}$ is the arc contained in the square S drawn in blue on the right-hand side of Figure 1. Such an annulus is a *product annulus* in the sense of [Juh08, Definition 2.9], and induces a sutured manifold decomposition

$$(N, \gamma_1) \xrightarrow{A} (M_0, \gamma_0).$$

By the work of Honda, Kazez, and Matić [HKM08, Section 7], the annulus A induces a gluing map

$$\Phi_\xi: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1).$$

By [Juh08, Lemma 8.9], if $H_2(N) = 0$, then the product annulus decomposition induces an isomorphism between $\text{SFH}(N, \gamma_1)$ and $\text{SFH}(M_0, \gamma_0)$. Looking at the proof of [Juh08, Lemma 8.9], we see that the technical assumption $H_2(N) = 0$ ensures that there is an *admissible* surface diagram \mathcal{H} adapted to the product annulus A such that $\text{CF}(\mathcal{H}, \mathfrak{s}) = 0$ for all $\mathfrak{s} \notin O_A$, where O_A is the set of outer Spin^c -structures. Although $H_2(N) \neq 0$ in our case, we will prove that the Heegaard diagram arising from a projection via the Ozsváth-Szabó construction satisfies the condition above.

Consider projections of (L_0, P_0) and (L_1, P_1) that locally look like in Figure 1, and construct a 2ℓ -pointed Heegaard diagram $\mathcal{H}_0 = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ for (L_0, P_0) as explained in Section 3.3. Let $\Pi_{\mathbf{x}}^*(\mathcal{H}_0)$ denote the set of periodic domains in \mathcal{H}_0 based at \mathbf{x} that *do not cross* the basepoints \mathbf{w} and \mathbf{z} . We now identify all such domains in the Heegaard diagram \mathcal{H}_0 . Since $\Pi_{\mathbf{x}}^*(\mathcal{H}_0)$ is an affine space over $H_2(M_0)$, it follows that

$$\Pi_{\mathbf{x}}^*(\mathcal{H}_0) \cong \mathbb{Z}^{\oplus(|L_0|-1)}.$$

The space $\Pi_{\mathbf{x}}^*(\mathcal{H}_0)$ is generated by the periodic domains associated to each link component, except the one that does not contain a ladybug. The generator in the case of the Hopf link is shaded grey on the right-hand side of Figure 4. One can make \mathcal{H}_0 admissible by performing finger moves on the ladybugs.

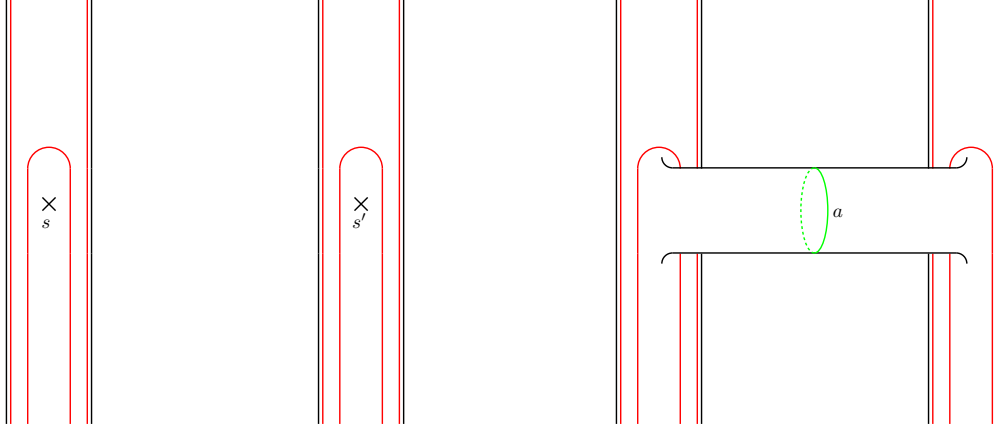


FIGURE 5. A Heegaard diagram for the link L_0 in a neighbourhood of the square S where the 2-handle is attached is depicted on the left-hand side. The Heegaard diagram on the right for the sutured manifold (N, γ_1) after the boundary cobordism is obtained by connecting the sutures s and s' by a tube. The core of the tube is the green curve a .

A Heegaard diagram \mathcal{H}_N for (N, γ_1) is obtained from the Heegaard diagram \mathcal{H}_0 simply by connecting two sutures (or basepoints) with an annulus; see Figure 5. The core a of the connecting annulus – shown in green in the picture – spans the decomposing annulus A in N .

Let $s, s' \subset s(\gamma_0)$ be the sutures on the left-hand side of Figure 5 corresponding to the decorations $p, p' \in P_0$ shown in red on the left-hand side of Figure 1. We claim that if we glue product 2-handles to M_0 along s and s' , corresponding to filling the sutures s and s' with disks in the diagram \mathcal{H}_0 , then the resulting sutured manifold (M'_0, γ'_0) is still balanced. The only thing we need to check is that each component of $\partial M'_0$ has a suture.

First, suppose that p and p' belong to different components of L_0 ; i.e., we have a merge saddle. Then, by definition, P_0 has at least two decorations on each component of L_0 . Hence gluing product 2-handles to s and s' results in two S^2 components of ∂M_0 with at least one suture on each.

Now suppose that p and p' lie on the same component of L_0 . Then there are two subcases depending on the orientability of the saddle. In case of an orientable saddle, we have $|L_1| = |L_0| + 1$. As the components of L_1 involved in the saddle move each have at least two decorations of P_1 , the corresponding component of L_0 has at least six points of P_0 (including p and p'). Furthermore, p and p' cannot be consecutive decorations, as that would result in a component of L_1 with no decorations. Hence ∂M_0 has two S^2 components with at least one suture on each. In case of an unorientable saddle, we have $|L_0| = |L_1|$. As the relevant component of L_1 has at least two decorations, the component of L_0 containing p and p' has at least four decorations. Furthermore, p and p' are not consecutive, and the result follows as above.

It follows from the above discussion that if we fill the sutures s and s' , then the resulting diagram \mathcal{H}'_0 can be made admissible by winding the β -curves. Furthermore,

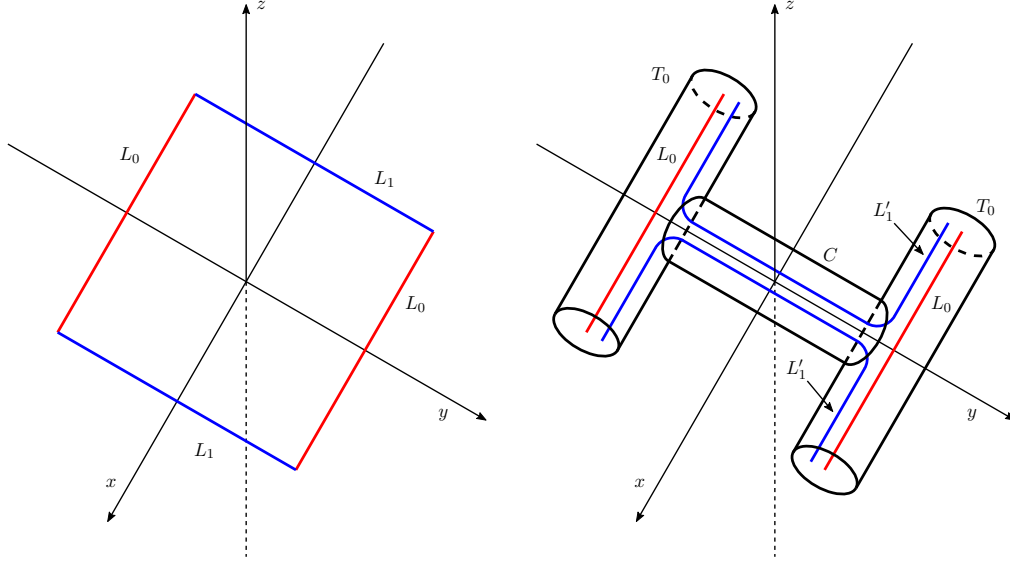


FIGURE 6. The left-hand side shows the square S embedded in S^3 . The tubular neighbourhood T_0 of L_0 is illustrated on the right, together with the cylinder C . Note that a link L'_1 isotopic to L_1 is contained in $T_0 \cup C$.

we can assume the winding happens away from the visible halves of the two tubes on the left-hand side of Figure 5. Without loss of generality, we can assume that \mathcal{H}_0 is already such that \mathcal{H}'_0 is admissible. Then we claim that \mathcal{H}_N is also admissible. Indeed, if P is a non-zero periodic domain in \mathcal{H}_N , then compressing it along the green curve a , we obtain a periodic domain P' in \mathcal{H}'_0 . As \mathcal{H}'_0 is admissible, P' , and hence also P has both positive and negative multiplicities.

In particular, we can assume that both diagrams \mathcal{H}_0 and \mathcal{H}_N are admissible. Furthermore, \mathcal{H}_N is adapted to A , and the quasi-polygon P that defines the annulus A is a regular neighbourhood of the green curve a on the right-hand side of Figure 5. Since P is disjoint from all α - and β -curves, we can apply the product annulus decomposition formula [Juh08, Lemma 8.9] and [HKM08, Proposition 6.4] to obtain that

$$\Phi_\xi: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1)$$

is an isomorphism.

3.4.2. The special cobordism associated to a saddle. Let us now focus our attention on the special cobordism \mathcal{W}^s associated to a saddle. We prove that such a cobordism consists of a single 2-handle attachment.

Consider the surface F embedded in $S^3 \times I$. Call t the coordinate on the I factor. The section $t = 0$ of the cobordism \mathcal{W}^s is M_0 , and the section $t = 1$ is M_1 . Notice that the cobordism \mathcal{W}^s is not a cobordism from (M_0, γ_0) to (M_1, γ_1) , but from (N, γ_1) to (M_1, γ_1) . The manifold N is the union of M_0 and the vertical boundary of a regular neighbourhood of F in $S^3 \times I$, which is diffeomorphic to $F \times S^1$.

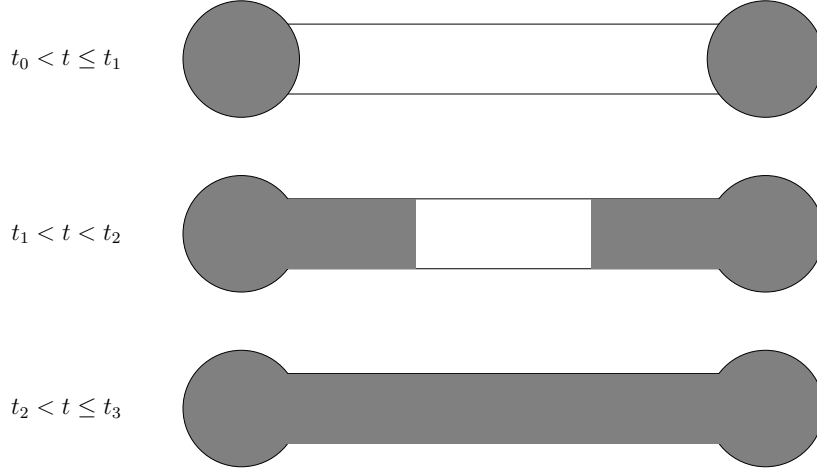


FIGURE 7. This figure shows how \mathcal{W}_t^s changes with t . The shaded area represents the planar section $\{x = 0\}$ of the complement of \mathcal{W}_t^s .

Choose local coordinates x , y , and z on S^3 , and suppose that the square in Figure 1 is $[-1, 1] \times [-1, 1] \times \{0\}$. Furthermore, suppose that $L_0 \cap S = [-1, 1] \times \{-1, 1\} \times \{0\}$; see the left-hand side of Figure 6. The manifold M_0 is the complement in $S^3 \times \{0\}$ of a radius ρ_0 tubular neighbourhood T_0 of L_0 for some $0 < \rho_0 \ll 1$. Let $0 < R_c < \rho_0$, and for every $t \in I$, define an open cylinder

$$C = C_t = \{ (x, y, z, t) : x^2 + z^2 < R_c^2, |y| < 1 \}$$

of radius R_c ; see the right-hand side of Figure 6.

For technical reasons that we will explain later, we also define tubular neighbourhoods T_t of L_0 for $t \in I$ that shrink as t increases, except in the intersection $T_0 \cap C$, where the tubular neighbourhood does not shrink. Choose a smooth decreasing function $t \mapsto \rho_t$ for $t \in I$, in such a way that $\rho_1 > R_c$, and let T_t be the tubular neighbourhood of L_0 of radius ρ_t , except that we require that $T_t \cap C = T_0 \cap C$.

Let $t_0 = 0$, $t_1 = 1/4$, $t_2 = 1/2$, $t_3 = 3/4$, and $t_4 = 1$. We can suppose that the t -sections \mathcal{W}_t^s of the cobordism \mathcal{W}^s appear as follows:

- For $t \in [t_0, t_1]$, we have $\mathcal{W}_t^s = S^3 \setminus T_t$. In particular $\mathcal{W}_0^s = M_0$.
- For $t \in [t_1, t_2]$, if we let

$$d_t = \{ (x, y, z, t) \in C_t : |y| > 2 - 4t \},$$

then $\mathcal{W}_t^s = S^3 \setminus (T_t \cup d_t)$. Note that we have $\mathcal{W}_{t_1}^s = S^3 \setminus T_{t_1}$; see Figure 7.

- For $t \in (t_2, t_3)$, we assume that $\mathcal{W}_t^s = S^3 \setminus (T_t \cup C_t)$; see Figure 7.
- For any $t, \bar{t} \in [t_3, t_4]$ such that $\bar{t} < t$, we have $\mathcal{W}_{\bar{t}}^s \subsetneq \mathcal{W}_t^s$. Furthermore, \mathcal{W}_t^s is the complement of a regular neighborhood of L_1 in S^3 that shrinks as t increases. We can assume this because, up to isotopy, we can suppose that L_1 is contained in $T_1 \cup C_t$; see the right-hand side of Figure 6.

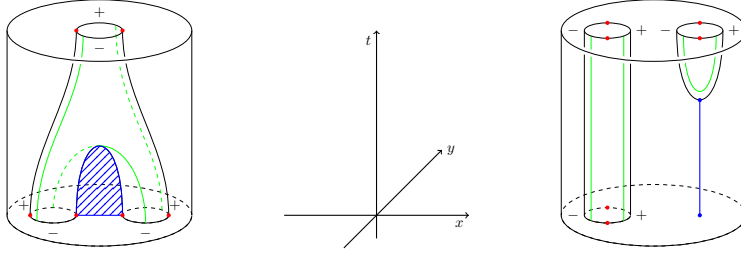


FIGURE 8. The figure above illustrates two decorated cobordisms. In both cases, the decorated links are embedded in S^3 , and the cobordism is embedded in $S^3 \times I$. The figure represents the $\{z = 0\}$ section $\mathbb{R}^2 \times \{0\} \times I$ of $\mathbb{R}^3 \times I \subseteq S^3 \times I$, with coordinates as shown above. Here, the coordinates on \mathbb{R}^3 are x , y , and z , and the coordinate on I is t . We use this convention every time a (decorated) cobordism in $S^3 \times I$ is represented enclosed in a cylinder, as above. The figure on the left shows a merge saddle cobordism from (U_2, P_{U_2}) – the two-component unlink with standard decorations – to (U_1, P_{U_1}) – the unknot with standard decorations. The core of the 2-handle described in Section 3.4.2 is shown in blue. The picture on the right shows a birth cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) . The core of the 1-handle described in Section 3.5 is shown in blue.

Let $c_t := C_t \cap \mathcal{W}^s$ for $t \leq t_2$ and $c_t = \emptyset$ otherwise. We define the following 0-codimensional submanifold of \mathcal{W}^s :

$$\mathfrak{H} = \{ (x, y, z, t) \in \mathcal{W}^s : (x, y, z) \in c_t \}.$$

Furthermore, let $\mathcal{N} \approx N \times I$ be a collar neighbourhood of N in \mathcal{W}^s .

Lemma 3.9. *The manifold $\mathcal{N} \cup \mathfrak{H}$ is a 2-handle cobordism from (N, γ_1) to a sutured manifold (\bar{N}, γ_1) . The attaching circle \mathbb{L} of the 2-handle is given by*

$$\begin{aligned} \mathbb{L} \cap M_0 &= \{ (0, y, 0, 0) : |y| \leq 1 - \rho_0 \}, \\ \mathbb{L} \cap \mathcal{W}_t^s &= \{ (0, \pm \min \{ 2 - 4t, 1 - \rho_0 \}, 0, t) \} \quad \text{if } t \in (0, t_2]. \end{aligned}$$

The framing f of the attaching sphere \mathbb{L} is given by

$$\begin{aligned} f \cap M_0 &= \left\{ (0, y, \varepsilon, 0) : |y| \leq 1 - \sqrt{\rho_0^2 - \varepsilon^2} \right\}, \\ f \cap \mathcal{W}_t^s &= \left\{ \left(0, \pm \min \left\{ 2 - 4t, 1 - \sqrt{\rho_0^2 - \varepsilon^2} \right\}, \varepsilon, t \right) \right\} \quad \text{if } t \in (0, t_2] \end{aligned}$$

for $\varepsilon \ll 1$.

Proof. For each x' and z' such that $(x')^2 + (z')^2 \leq R_c^2$, define the disc

$$D_{x', z'} := \mathfrak{H} \cap \{x = x'\} \cap \{z = z'\}.$$

Note that $D_{x', z'}$ is properly embedded in \mathcal{W}^s , in the sense that

$$\partial D_{x', z'} = D_{x', z'} \cap \partial \mathcal{W}^s.$$

The discs $D_{x',z'}$ are disjoint from each other and give a foliation of \mathfrak{H} . By reparametrising each $D_{x',z'}$, we can give a 2-handle structure to \mathfrak{H} :

$$\mathfrak{H} \approx D_{0,0} \times \{ (x', z') : (x')^2 + (z')^2 \leq R_c^2 \}.$$

The disc $D_{0,0}$, illustrated on the left-hand side of Figure 8, is the core of \mathfrak{H} , and its attaching sphere is $\mathbb{L} := \partial D_{0,0}$, which is precisely the curve described in the statement of this lemma. The framing of the \mathfrak{H} is $f := \partial D_{0,\varepsilon}$ for some small ε . \square

Consider now the special cobordism $\overline{\mathcal{W}^s} := \overline{\mathcal{W}^s} \setminus \mathfrak{H}$ from the sutured manifold (\overline{N}, γ_1) , which is the result of the surgery on (N, γ_1) along (\mathbb{L}, f) , to (M_1, γ_1) .

Lemma 3.10. *The cobordism $\overline{\mathcal{W}^s}$ from \overline{N} to M_1 is diffeomorphic to a product.*

Proof. Consider the function

$$t: \overline{\mathcal{W}^s} \rightarrow \mathbb{R}$$

that sends every point of $\overline{\mathcal{W}^s}$ to its t -coordinate. The flow-lines of $\partial/\partial t$ are vertical lines. A key property of $\overline{\mathcal{W}^s}$ is that every vertical line $\{ (x_0, y_0, z_0, t) : t \in I \}$ intersects $\overline{\mathcal{W}^s}$ in a single segment s_{x_0, y_0, z_0} with one endpoint in \overline{N} and the other endpoint in M_1 . The fact that the tubular neighbourhood T_t shrinks as t increases implies that s_{x_0, y_0, z_0} is properly embedded, in the sense that

$$(2) \quad \partial s_{x_0, y_0, z_0} = s_{x_0, y_0, z_0} \cap \partial \overline{\mathcal{W}^s}.$$

Actually, according to our definition, c_t does not shrink as t increases, so the flow-lines of $\partial/\partial t$ emanating from c_0 do not satisfy Equation (2). However, via an isotopy, we can achieve that the sections of $(S^3 \times I) \setminus \overline{\mathcal{W}^s}$ shrink as t increases. It follows that we can assume that the flow lines of $\partial/\partial t$ connect \overline{N} with M_1 .

We would like to conclude that $\overline{\mathcal{W}^s}$ is diffeomorphic to a product cobordism from (\overline{N}, γ_1) to (M_1, γ_1) , but we first need to fix two technical problems. The first issue is that $\overline{\mathcal{W}^s}$ is not quite a sutured manifold cobordism, as $\partial \overline{N} = \partial M_1$. This can be fixed by stacking the product cobordism $\partial M_1 \times [1, 2]$ on top of $\overline{\mathcal{W}^s}$ and smoothing along ∂M_0 ; see Figure 9. We set the vertical part of $\overline{\mathcal{W}^s}$ to be $Z = \partial M_1 \times [1, 2]$. Finally, we rescale the function t to map to $[0, 1]$.

The second issue is that the function t is not a Morse function on the cobordism $\overline{\mathcal{W}^s}$ as $t|_{\overline{N}}$ is not constant. However, we can rescale the function t to obtain a new function \tilde{t} such that $\tilde{t}|_{\overline{N}} \equiv 0$, $\tilde{t}|_{M_1} \equiv 1$, and $\nabla \tilde{t}$ has the same direction as $\partial/\partial t$.

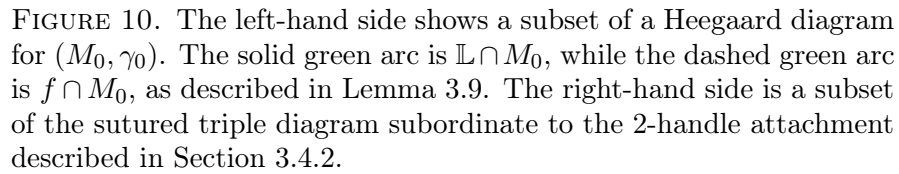
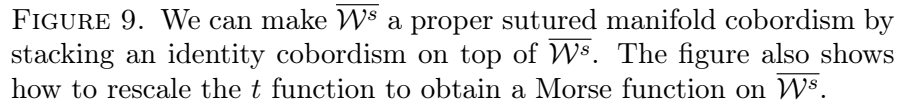
Specifically, since the sections of $(S^3 \times I) \setminus \overline{\mathcal{W}^s}$ shrink, we can assume that \overline{N} is the graph of a function

$$f: (x, y, z) \mapsto t(s_{x, y, z} \cap \overline{N});$$

see Figure 9 for an illustration. Furthermore, we can choose the shrinking of the sections of $(S^3 \times I) \setminus \overline{\mathcal{W}^s}$ in such a way that f is smooth. We define the function \tilde{t} as

$$(3) \quad \tilde{t}(x, y, z, t) := \frac{t - f(x, y, z)}{1 - f(x, y, z)}.$$

Then \tilde{t} is a Morse function on $\overline{\mathcal{W}^s}$ without any critical points, and therefore $\overline{\mathcal{W}^s}$ is diffeomorphic to a product cobordism. \square



3.4.3. *A Heegaard diagrammatic description.* Let \mathcal{X} be a saddle decorated link cobordism. We now give a description of the cobordism map

$$F_{\mathcal{X}}: \widehat{\text{HFL}}(L_0, P_0) \rightarrow \widehat{\text{HFL}}(L_1, P_1)$$

on the level of Heegaard diagrams. Recall that such a map is the composition of two maps:

$$\Phi_{\xi}: \widehat{\text{HFL}}(L_0, P_0) \cong \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1),$$

called the gluing map, and

$$F_{\mathcal{W}^s}: \text{SFH}(N, \gamma_1) \rightarrow \text{SFH}(M_1, \gamma_1) \cong \widehat{\text{HFL}}(L_1, P_1),$$

corresponding to the special cobordism.

First, we use a connected projection of L_0 to find a Heegaard diagram

$$\mathcal{H}_0 = (\Sigma_0, \alpha_0, \beta_0, z_0, w_0)$$

for (M_0, γ_0) as explained in Section 3.3. Recall that the Heegaard diagram locally looks like the left-hand side of Figure 5. The decorations on the projection yield basepoints (or sutures) on the Heegaard diagram, and the submanifold $R_-(P_0) \subseteq L_0$ corresponds to the ladybugs (except for one component). Note that the construction of Ozsváth and Szabó for the Heegaard diagram is explicit and the surface Σ_0 arising from a link projection naturally sits in S^3 . In particular, the solid green arc and the dashed green arc on the left-hand side of Figure 10 are arcs in S^3 . The solid green arc is $\mathbb{L} \cap M_0$, and the dashed green arc is the framing $f \cap M_0$, as in Lemma 3.9.

The effect of the gluing map Φ_{ξ} was described in Section 3.4.1: The Heegaard diagram is modified by gluing an annulus B as in Figure 5, and the map on the chain level is given by the evident bijection between the generators of the Floer complexes. The sutured manifold described by such a Heegaard diagram is $N = M_0 \cup (-Z)$. The submanifold $M_0 \subseteq N$ is given by the Heegaard diagram before attaching the annulus B , while the submanifold $-Z \subseteq N$ is given in the Heegaard diagram by the annulus B .

We now turn to the special cobordism \mathcal{W}^s , and we assume that it is parameterised as in Section 3.4.2. With respect to this parameterisation, the annulus B is embedded in N in the section $\{x = 0\}$. We can actually give an explicit description of B : For $t \in [t_0, t_3]$, the subset $B \cap \mathcal{W}_t^s$ is the boundary of the shaded region in Figure 7. It follows that $B \cap [t_0, t_3]$ is a pair-of-pants B_1 . Furthermore, $B \cap \mathcal{W}_{t_3}^s$ is a circle that bounds a disc B_2 in $N \cap \{t \geq t_3\}$, as we now explain: If one considers the right-hand side of Figure 6, for each $t \geq t_3$ there is a disc D_t in $T_t \cup C_t$ that is disjoint from L'_1 , obtained by intersecting $T_t \cup C_t$ with the $\{x = 0\}$ section. As $t \geq t_3$ increases, the disc shrinks to a point and then disappears. By taking the union of the boundaries ∂D_t for $t \geq t_3$, one gets the disc B_2 . The annulus B is the union $B_1 \cup B_2$.

Our next aim is to construct a Heegaard diagram subordinate to the 2-handle with attaching sphere \mathbb{L} and framing f ; cf. [Juh09, Definition 6.3], where \mathbb{L} and f are as defined in Lemma 3.9. As we already noted, $\mathbb{L} \cap M_0$ and $f \cap M_0$ are the solid and the dashed green arcs shown on the left-hand side of Figure 10. The intersections $\mathbb{L} \cap (-Z)$ and $f \cap (-Z)$ are contained inside B , because they both sit in the $\{x = 0\}$ section of N . They determine two parallel arcs $\mathbb{L} \cap B$ and $f \cap B$ around the annulus B .

In Figure 5, B is an *abstract* annulus in the Heegaard diagram for (N, γ_1) ; i.e., we are not given an embedding of B into $S^3 \times I$. Hence, we can assume that $\mathbb{L} \cap B$ and $f \cap B$ are straight segments joining the two boundary components of B (that is, they do not wind around the annulus).

We claim that the Heegaard diagram on the right-hand side of Figure 10 is subordinate to the 2-handle attachment along \mathbb{L} with framing f , in the sense of [Juh09, Definition 6.3]. The δ -curves not shown in the figure are small Hamiltonian translates of the β -curves.

Consider the Heegaard diagram on the right-hand side of Figure 5, and suppose that it is embedded in S^3 as in the figure. Let $\Sigma \times I$ be a regular neighborhood of the Heegaard surface Σ in S^3 . The translates $\Sigma_+ = \Sigma \times \{1\}$ and $\Sigma_- = \Sigma \times \{0\}$ of Σ appear on the inside and on the outside of $\Sigma = \Sigma \times \{0.5\}$, respectively. We denote the translates of the annulus $B \subset \Sigma$ by $B_+ \subset \Sigma_+$ and $B_- \subset \Sigma_-$.

Recall that the attaching circle \mathbb{L} consists of two arcs, one of which is contained in B , and the other one in M_0 . Let $B(\mathbb{L})$ be the bouquet obtained by connecting \mathbb{L} to B_+ via a straight arc entirely contained in $B \times I$.

Consider now the Heegaard diagram on the right-hand side of Figure 10, which again we can think of as embedded in S^3 , and denote it by (Σ', α, β) . In order to construct the sutured manifold associated to it, one should take $\Sigma' \times I$, and attach thickened discs to Σ'_- along the α -curves and to Σ'_+ along the β -curves. We denote these discs by $D_{\alpha_i}^2$ and $D_{\beta_i}^2$. Given subsets $\alpha' \subset \alpha$ and $\beta' \subset \beta$, let $\Sigma'(\alpha', \beta')$ be the sutured manifold obtained from $\Sigma' \times I$ by attaching the thickened discs corresponding to the α - and β -curves in α' and β' . The manifold $\Sigma' \times I$ can also be viewed as a subset of S^3 , as well as the disc $D_{\alpha_1}^2$, which lies in the plane of the link projection as it arose from the Ozsváth-Szabó diagram explained in Subsection 3.3, so it follows that $\Sigma'(\alpha_1)$ is also embedded in S^3 .

The manifold $\Sigma'(\alpha_1, \alpha_2)$ is obtained from $\Sigma'(\alpha_1)$ by attaching the thickened disc $D_{\alpha_2}^2$. The upper half of α_2 is parallel to the upper half of α_1 along the top of Figure 10 and outside of it, hence we can slide the upper half of $D_{\alpha_2}^2$ along $D_{\alpha_1}^2$ until the upper boundary becomes a horizontal arc across $D_{\alpha_1}^2$, and $D_{\alpha_2}^2$ becomes perpendicular to $D_{\alpha_1}^2$. After the isotopy, $D_{\alpha_2}^2$ connects the small handle in the centre of Figure 10 to $D_{\alpha_1}^2 \times I$. This implies that $\Sigma'(\alpha_1, \alpha_2)$ can also be seen as embedded in S^3 , and we can isotope the small handle in the centre of Figure 10 onto $D_{\alpha_1}^2$. Attaching all the other handles along the α - and β -curves except β_1 , one obtains $\Sigma'(\alpha_1, \dots, \alpha_g, \beta_2, \dots, \beta_g)$, which is the same manifold as the one defined by the Heegaard diagram on the right-hand side of Figure 5 – that is, N – with the difference that a neighbourhood of $B(\mathbb{L})$ has been removed from it. The curve β_1 is the meridian of \mathbb{L} , and δ_1 appears exactly as the dashed arc on the left-hand side of Figure 10. Therefore, this is a Heegaard diagram subordinate to the 2-handle attachment.

By Lemmas 3.9 and 3.10, the cobordism \mathcal{W}^s consists just of this 2-handle attachment, and the map $\mathcal{F}_{\mathcal{W}^s}$ is given by counting clockwise holomorphic triangles in the triple Heegaard diagram on the right-hand side of Figure 10.

3.4.4. Saddle cobordisms and the decorated surgery exact triangle. In this subsection, we use the Heegaard diagrammatic description of the saddle cobordism to show that

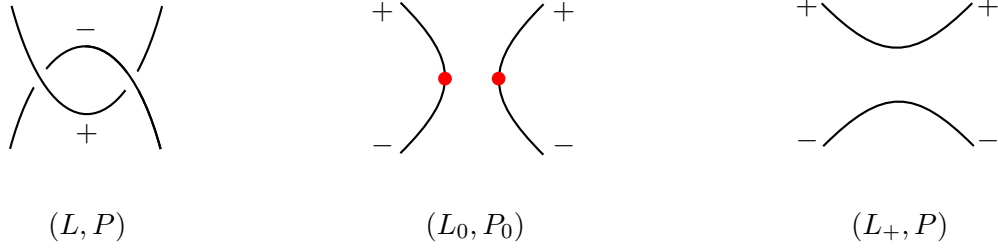


FIGURE 11. Suppose that the decorated links (L_0, P_0) and (L_+, P) are related by a saddle cobordism, and we obtain (L, P) from (L_+, P) by adding a full right-handed twist. The links (L, P) , (L_0, P_0) , and (L_+, P) fit into the decorated skein exact triangle of Theorem 2.18, and the map from $\widehat{\text{HFL}}(L_0, P_0)$ to $\widehat{\text{HFL}}(L_+, P)$ in this triangle agrees with the saddle cobordism map.

the induced map agrees with the map

$$f: \widehat{\text{HFL}}(L_0, P_0) \rightarrow \widehat{\text{HFL}}(L_+, P)$$

in the decorated version of the Ozsváth-Szabó skein exact triangle that we stated in Theorem 2.18.

Suppose that (L_0, P_0) and (L_+, P) are links in S^3 that are related by a saddle cobordism as in Figure 1. Let L be the link obtained from L_+ by adding a full right-handed twist. Then (L, P) , (L_0, P_0) , and (L_+, P) have projections that differ only locally as in Figure 11, and hence fit into the surgery exact triangle

$$\longrightarrow \widehat{\text{HFL}}(L, P) \xrightarrow{e} \widehat{\text{HFL}}(L_0, P_0) \xrightarrow{f} \widehat{\text{HFL}}(L_+, P) \longrightarrow$$

of Theorem 2.18. Indeed, consider the disk D that is perpendicular to the plane of Figure 11 and intersects it in a horizontal arc that meets L in two points. Then resolving L along D gives L_0 , and doing a right twist gives L_+ .

Theorem 3.11. *Let the triple (L, P) , (L_0, P_0) , (L_+, P) be as in Figure 11. Then the map*

$$f: \widehat{\text{HFL}}(L_0, P_0) \longrightarrow \widehat{\text{HFL}}(L_+, P)$$

appearing in the decorated skein exact triangle of Theorem 2.18 coincides with the cobordism map induced by the saddle cobordism from (L_0, P_0) to (L_+, P) .

Proof. Consider the saddle cobordism \mathcal{S} from (L_0, P_0) to (L_+, P) that can be represented as in Figure 1. Then the cobordism map

$$F_{\mathcal{S}}: \widehat{\text{HFL}}(L_0, P_0) \rightarrow \widehat{\text{HFL}}(L_+, P)$$

is given by the isomorphism induced by gluing along a product annulus as in Section 3.4.1, followed by the map obtained by counting clockwise holomorphic triangles in the triple diagram $(\Sigma, \alpha, \beta, \delta)$ in Figure 10.

Consider the crossing disk D that intersects L in two points, and let $K = \partial D$. As in Section 2.3, the framings $m = \infty$, $l = 0$, and $s = 1$ of K give rise to the surgery

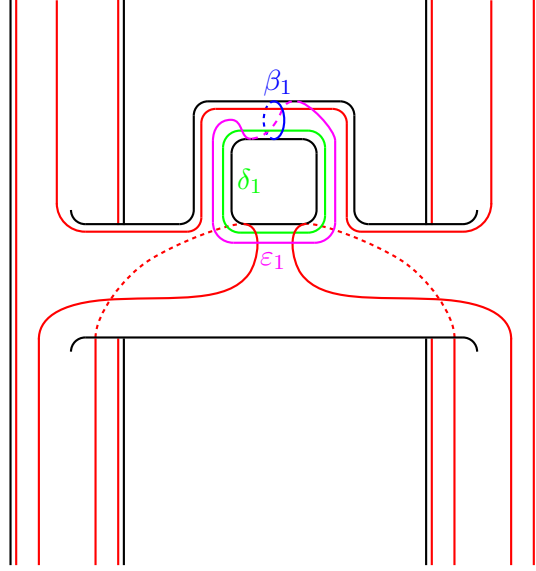


FIGURE 12. The quadruple diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ above has four sets of attaching curves. The triple diagram $(\Sigma, \alpha, \beta, \delta)$ is subordinate to the 2-handle attachment described in Section 3.4.2. The diagram $(\Sigma, \alpha, \varepsilon)$ is for $S^3(L, P) = (M_\infty(K), \gamma)$, while (Σ, α, β) is for $(M_0(K), \gamma)$, and (Σ, α, δ) is for $S^3(L_+, P) = (M_1(K), \gamma)$. The δ -curves and the ε -curves not shown are small isotopic translates of the β -curves.

exact triangle

$$(4) \quad \dots \longrightarrow \mathrm{SFH}(M_\infty(K), \gamma) \xrightarrow{e'} \mathrm{SFH}(M_0(K), \gamma) \xrightarrow{f'} \mathrm{SFH}(M_1(K), \gamma) \longrightarrow \dots$$

If A denotes the product annulus obtained from D in $M_0(K)$, and

$$\Phi_A: \widehat{\mathrm{HFL}}(L_0, P_0) \rightarrow \mathrm{SFH}(M_0(K), \gamma)$$

is the gluing map, then we obtain the exact triangle

$$\dots \longrightarrow \widehat{\mathrm{HFL}}(L, P) \xrightarrow{e} \widehat{\mathrm{HFL}}(L_0, P_0) \xrightarrow{f} \widehat{\mathrm{HFL}}(L_+, P) \longrightarrow \dots$$

by setting $e = \Phi_A^{-1} \circ e'$ and $f = f' \circ \Phi_A$.

The result follows once we show that the quadruple diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ in Figure 12 is as in the statement of [OS04, Lemma 9.2], and counting holomorphic triangles in such a diagram yields the exact sequence (4).

Consider a local projection for L as on the left of Figure 13, and let K be the boundary of the crossing disk D . Let μ and μ' be two concentric meridians of K . We view μ as a knot with framing μ' . After performing 0-surgery along K , the framed knot (μ, μ') becomes isotopic to the core of the solid torus that we glued in during the surgery, and which we denote by (μ_*, μ'_*) in $S^1 \times S^2 = S_0^3(K)$; i.e., μ_* is $S^1 \times \{\mathrm{pt}\}$ and μ'_* is $S^1 \times \{\mathrm{pt}'\}$. These are shown in green in the middle of Figure 13. This

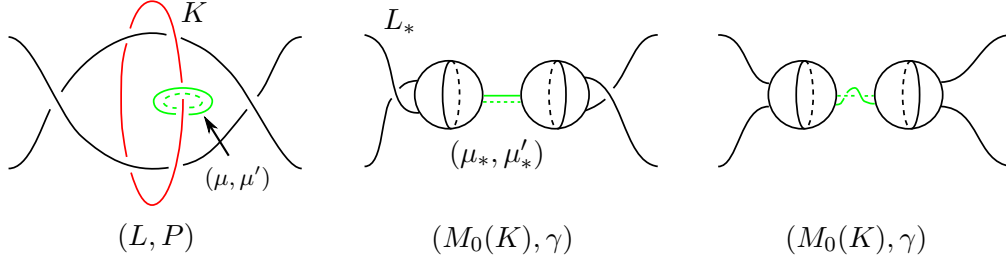


FIGURE 13. The left-hand side of the figure shows the link (L, P) with the small unknot $K = \partial D$ in red, and the two concentric meridians of K , denoted by μ and μ' , in green. In the middle, we see the result of 0-surgery along K . The manifold $S^3_0(K) = S^1 \times S^2$ is represented here as the complement of two balls, with their boundaries identified via a reflection. The framed knot (μ_*, μ'_*) is the image of (μ, μ') . After untwisting L^* , we get the figure on the right, making the solid and the dashed green arcs linked.

means that performing surgery along the framed knot (μ_*, μ'_*) in $(M_0(K), \gamma)$, we recover $S^3(L, P) = (M_\infty(K), \gamma)$. In particular, $M_0(K) \setminus N(\mu_*) = M_\infty(K) \setminus N(K)$.

The disc D can be capped off in $M_0(K) \subset S^1 \times S^2$ to obtain a sphere $S^2 \subset M_0(K)$. If we cut $S^1 \times S^2$ along this sphere, the manifold that we obtain is the complement of the two balls in the middle of Figure 13. To recover $S^1 \times S^2$, glue the boundaries of the balls via an identification given by a reflection. Note that, in the middle of Figure 13, the framed knot (μ_*, μ'_*) appears as two parallel arcs.

Performing 0-surgery on K sends the link L to the link L_* in the middle of Figure 13. By twisting the left-hand sphere by π to the left and the right-hand sphere by π to the right along the horizontal axis (this is an automorphism of $S^1 \times S^2$), we remove the two crossings of L , but introduce a full right-handed twist in the green pair of arcs μ_* and μ'_* ; see the right-hand side of Figure 13. By the twist, the link L_* is sent to the link on the right-hand side of Figure 13, whose complement has a Heegaard diagram as on the right of Figure 5.

Recall from Section 3.4.3 that the triple diagram $(\Sigma, \alpha, \beta, \delta)$ is subordinate to performing surgery on $(M_0(K), \gamma)$ along the framed knot \mathbb{L} shown on the left of Figure 10. On the right of Figure 13, the knot \mathbb{L} corresponds to μ_* with a framing f that is a parallel vertical copy of μ_* . Hence, surgery on $(M_0(K), \gamma)$ along (μ_*, μ'_*) gives $(M_\infty(K), \gamma)$, while surgery along (μ_*, f) gives $(M_1(K), \gamma)$.

Let $B(\mu_*)$ be a bouquet for the knots μ_* in $(M_\infty(K), \gamma)$, and let $B(K)$ be the corresponding bouquet of K in $(M_0(K), \gamma)$, as defined in [Juh09]. The Heegaard subdiagram $(\Sigma, \alpha, \beta \setminus \beta_1)$ of the diagram on the right-hand side of Figure 10 represents the sutured manifold $(M_0(K) \setminus B(\mu_*), \gamma)$, or, equivalently, $(M_\infty(K) \setminus B(K), \gamma)$. The curve β_1 on the right-hand side of Figure 10 corresponds to a meridian of μ_* , so Dehn filling the manifold described by $(\Sigma, \alpha, \beta \setminus \beta_1)$ along β_1 yields $(M_0(K), \gamma)$. The curve δ_1 corresponds to f , so Dehn filling along it yields $S^3(L_+) = (M_1(K), \gamma)$. The curve ε_1 in Figure 12 corresponds to μ'_* , and so filling along it yields $S^3(L) = (M_\infty(K), \gamma)$. Furthermore, it satisfies the condition $[\beta_1] + [\delta_1] = [\varepsilon_1]$ in the first homology of

the boundary, and β_1 , δ_1 , and ε_1 are oriented such that their pairwise algebraic intersection numbers are -1 . It follows that the Heegaard diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ satisfies the hypotheses of [OS04, Lemma 9.2], and therefore the count of holomorphic triangles in such a diagram gives the exact triangle of Theorem 2.18. \square

3.5. The map associated to a birth cobordism. We now turn to the description of the map associated to a birth cobordism $\mathcal{B} = (F, \sigma)$ from (L, P) to $(L \sqcup U_1, P \sqcup P_{U_1})$, where (U_1, P_{U_1}) is the unknot with two decorations. Let $(M_0, \gamma_0) = \mathcal{W}(L, P)$ and $(M_1, \gamma_1) = \mathcal{W}(L \sqcup U_1, P \sqcup P_{U_1})$ denote the sutured manifolds complementary to the links. As usual, we write the cobordism $\mathcal{W}(\mathcal{B})$ as the composition of a boundary cobordism \mathcal{W}^b from (M_0, γ_0) to (N, γ_1) , and a special cobordism \mathcal{W}^s from (N, γ_1) to (M_1, γ_1) , where

$$(N, \gamma_1) = (M_0, \gamma_0) \sqcup (S^1 \times D^2, \gamma),$$

and γ consists of two longitudes as sutures. Note that $\text{SFH}(S^1 \times D^2, \gamma) \cong \mathbb{F}_2$, and hence

$$\text{SFH}(M_0, \gamma_0) \cong \text{SFH}(M_0, \gamma_0) \otimes \mathbb{F}_2 \cong \text{SFH}(N, \gamma_1),$$

and that the gluing map is the above isomorphism.

The manifold (M_1, γ_1) is the connected sum of (M_0, γ_0) and the complement of the unknot, which is $(S^1 \times D^2, \gamma)$, so we have that

$$\text{SFH}(M_1, \gamma_1) \cong \text{SFH}(M_0, \gamma_0) \otimes V,$$

where $V = \mathbb{F}_2 \langle B, T \rangle$ is the 2-dimensional vector field generated by two homogeneous vectors B (bottom-graded) and T (top-graded).

In the following lemma, we prove that \mathcal{W}^s consists of a single 1-handle attachment. The 1-handle of the special cobordism \mathcal{W}^s is illustrated on the right-hand side of Figure 8 in the case of a birth cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) .

Lemma 3.12. *The special cobordism \mathcal{W}^s associated to a birth consists of the 1-handle attachment that connects the two components of N .*

Proof. Suppose that the birth of the new component is centred (with respect to some local coordinate system on S^3) at $O = (0, 0, 0)$. Without loss of generality, we can assume that the birth takes place in a ball $B_R(O)$ of radius R centered at the origin so small that it is disjoint from L .

We can suppose that the sections \mathcal{W}_t^s of the special cobordism appear as follows:

- for $t \in [0, 1/2]$, we have $\mathcal{W}_t^s = M_0$,
- for $t \in (1/2, 3/4)$, we have $\mathcal{W}_t^s = M_0 \setminus B_R(O)$,
- for $t \in [3/4, 1]$, since $U_1 \subseteq B_R(O)$, we can suppose that, if $t > \bar{t}$, then $\mathcal{W}_t^s \supseteq \mathcal{W}_{\bar{t}}^s$, and $\mathcal{W}_1^s = M_1$.

We define

$$\mathfrak{H} := \{ (x, y, z, t) : (x, y, z) \in B_R(O) \text{ and } t \in [0, 1/2] \}.$$

This is a 1-handle that connects the two components of N . The 1-handle structure is clear from the diffeomorphism

$$\mathfrak{H} \cong [0, 1/2] \times B_R(O) \cong D^1 \times D^3.$$

Call \overline{N} the result of the surgery on N , and let $\overline{\mathcal{W}}^s := \mathcal{W}^s \setminus \mathfrak{H}$. Then, in a very similar way to Lemma 3.10, we have that $\overline{\mathcal{W}}^s$ is a product cobordism from \overline{N} to M_1 . It follows that \mathcal{W}^s consists of a single 1-handle joining the two components of N . \square

Hence, by [Juh09, Definition 7.5], the map associated to the special cobordism is

$$F_{\mathcal{W}^s}: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(M_0, \gamma_0) \otimes V.$$

$$x \mapsto x \otimes T$$

Putting the above results together, we obtain the following.

Proposition 3.13. *Let \mathcal{B} be a birth cobordism from (L_0, P_0) to (L_1, P_1) . Then there is an isomorphism*

$$\widehat{\text{HFL}}(L_1, P_1) \cong \widehat{\text{HFL}}(L, P) \otimes V$$

such that $F_{\mathcal{B}}(x) = x \otimes T$ for every $x \in \widehat{\text{HFL}}(L_0, P_0)$.

3.6. The map associated to a death cobordism. Let \mathcal{D} be a death cobordism from $(L \sqcup U_1, P \sqcup P_{U_1})$ to (L, P) . This yields a cobordism

$$\mathcal{W} = \mathcal{W}(\mathcal{D}): S^3(L \sqcup U_1, P \sqcup P_{U_1}) \rightarrow S^3(L, P)$$

of sutured manifolds, where $\mathcal{W} = (W, Z, [\xi])$. Here $S^3(L \sqcup U_1, P \sqcup P_{U_1})$ and $S^3(L, P)$ denote the sutured manifolds complementary to the respective decorated links. For brevity, we will omit the decorations from the notation for the rest of this section.

We can turn the cobordism \mathcal{W} upside down [Juh09, Remark 2.13] and obtain a cobordism

$$\overline{\mathcal{W}} = (W, Z, [-\xi]): -S^3(L) \rightarrow -S^3(L \sqcup U_1).$$

Remark. Note that $\overline{\mathcal{W}}$ is diffeomorphic as an *oriented* sutured manifold cobordism to a birth cobordism

$$\mathcal{W}(\mathcal{B}): S^3(\overline{L}) \rightarrow S^3(\overline{L} \sqcup U_1) = S^3(\overline{L \sqcup U_1}),$$

where \overline{L} denotes the mirror image of the link L . Such a diffeomorphism is obtained by considering the reflection of the link L into the (x, y) -plane in S^3 , which extends to the cobordism. The reflection maps L to \overline{L} and reverses the orientation of S^3 . In this way, we can view $\overline{\mathcal{W}}$ as a birth cobordism.

As usual, we write the cobordism $\overline{\mathcal{W}}$ as a composition $\overline{\mathcal{W}} = \overline{\mathcal{W}}^s \circ \overline{\mathcal{W}}^b$, where $\overline{\mathcal{W}}^b$ is a boundary cobordism and $\overline{\mathcal{W}}^s$ is a special cobordism. Using the description of the birth cobordism maps in Section 3.5, we know that the following diagram is commutative:

$$\begin{array}{ccc} \text{SFH}(S^3(\overline{L})) & \xrightarrow{F_{\overline{\mathcal{W}}^b}} & \text{SFH}(S^3(\overline{L}) \sqcup (S^1 \times D^2, \gamma)) \\ & \searrow \text{id} \quad \quad \quad \downarrow \mathfrak{s} & \\ & & \text{SFH}(S^3(\overline{L})), \end{array}$$

where, $(S^1 \times D^2, \gamma)$ denotes a solid torus with two longitudinal sutures. As for the map associated to the special cobordism $\overline{\mathcal{W}}^s$, by Section 3.5, we know that the diagram

$$\begin{array}{ccc}
\mathrm{SFH}(S^3(\bar{L}) \sqcup (S^1 \times D^2, \gamma)) & \xrightarrow{F_{\bar{\mathcal{W}}^s}} & \mathrm{SFH}(S^3(\bar{L} \sqcup U_1)) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{SFH}(S^3(\bar{L})) & \longrightarrow & \mathrm{SFH}(S^3(\bar{L})) \otimes V \\
\mathbf{x} \mapsto & \longrightarrow & \mathbf{x} \otimes T
\end{array}$$

is commutative, where V is the 2-dimensional vector space over \mathbb{F}_2 generated by two homogeneous elements T (top-graded) and B (bottom-graded).

By turning the decomposition $\bar{\mathcal{W}} = \bar{\mathcal{W}}^s \circ \bar{\mathcal{W}}^b$ upside down, we obtain a decomposition of our original cobordism $\mathcal{W} = \mathcal{W}^b \circ \mathcal{W}^s$, where

- \mathcal{W}^s is the *special* cobordism obtained by turning $\bar{\mathcal{W}}^s$ upside down,
- \mathcal{W}^b is the (*not necessarily boundary*) cobordism obtained by turning $\bar{\mathcal{W}}^b$ upside down.

We study the maps induced by these in the following two subsections.

3.6.1. *The map associated to \mathcal{W}^s .* We consider the following commutative diagram:

$$\begin{array}{ccccc}
& \mathrm{SFH}(S^3(L)) \otimes V & \longrightarrow & \mathrm{SFH}(S^3(L)) & \\
& \downarrow \wr & & \downarrow \wr & \\
\varphi_1 \swarrow & \mathrm{SFH}(S^3(L \sqcup U_1)) & \xrightarrow{F_{\mathcal{W}^s}} & \mathrm{SFH}(S^3(L) \sqcup (\mathbb{S}, \gamma)) & \nwarrow \varphi_2 \\
& \downarrow \wr & & \downarrow \wr & \\
& \mathrm{SFH}(-S^3(L \sqcup U_1))^* & \xrightarrow{F_{\mathcal{W}^s}^*} & \mathrm{SFH}(-S^3(L) \sqcup -(\mathbb{S}, \gamma))^* & \\
& \downarrow \wr & & \downarrow \wr & \\
& \mathrm{SFH}(-S^3(L))^* \otimes V^* & \longrightarrow & \mathrm{SFH}(-S^3(L))^* & \\
& \downarrow \wr & & \downarrow \wr & \\
& \mathbf{x}^* \otimes T^* & \mapsto & \mathbf{x}^* & \\
& \downarrow \wr & & \downarrow \wr & \\
& \mathbf{x}^* \otimes B^* & \mapsto & 0 &
\end{array}$$

where $\mathbb{S} = S^1 \times D^2$. The commutativity of the central square follows from [Juh09, Theorem 11.9]. The map

$$\begin{aligned}
\varphi_2: \mathrm{SFH}(-S^3(L))^* &\rightarrow \mathrm{SFH}(S^3(L)) \\
\mathbf{x}^* &\mapsto \mathbf{x}
\end{aligned}$$

is the natural isomorphism explained in [Juh09, Section 11.2]: Given a Heegaard diagram $(\Sigma_2, \alpha_2, \beta_2)$ for $S^3(L)$, the diagram $(-\Sigma_2, \alpha_2, \beta_2)$ represents $-S^3(L)$, and

$$\mathrm{CF}(-\Sigma_2, \alpha_2, \beta_2)^* \cong \mathrm{CF}(\Sigma_2, \alpha_2, \beta_2)$$

by mapping \mathbf{x}^* to \mathbf{x} . Here $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a basis element of $\mathrm{CF}(\Sigma_2, \alpha_2, \beta_2)$, we also write \mathbf{x} for the corresponding basis element of $\mathrm{CF}(-\Sigma_2, \alpha_2, \beta_2)$, while \mathbf{x}^* denotes the respective dual basis element of $\mathrm{CF}(-\Sigma_2, \alpha_2, \beta_2)^*$.

The map φ_1 in the above diagram is less straightforward. We have

$$\begin{aligned}\varphi_1: \text{SFH}(-S^3(L))^* \otimes V^* &\rightarrow \text{SFH}(S^3(L)) \otimes V, \\ \mathbf{x}^* \otimes T^* &\mapsto \mathbf{x} \otimes B \\ \mathbf{x}^* \otimes B^* &\mapsto \mathbf{x} \otimes T\end{aligned}$$

where T and B respectively denote the top- and the bottom-graded intersection points of the α - and the β -curves contained in the “connected sum tube.” The reason for the apparent asymmetry in the map φ_1 is the following. If $(\Sigma_1, \alpha_1, \beta_1)$ and $(-\Sigma_1, \alpha_1, \beta_1)$ are Heegaard diagrams for $S^3(L \sqcup U_1)$ and $-S^3(L \sqcup U_1)$ respectively, then the two intersection points in the connected sum tube are denoted by T and B in one case and by B and T respectively in the other case. In both cases, we denote by T the top-graded intersection point. In V^* , the bottom-graded basis element is T^* , and the top-graded basis element is B^* , which makes φ_1 grading preserving. Therefore, we finally obtain that the following diagram is commutative:

$$\begin{array}{ccc}\text{SFH}(S^3(L \sqcup U_1)) & \xrightarrow{F_{\mathcal{W}^s}} & \text{SFH}(S^3(L) \sqcup (\mathbb{S}, \gamma)) \\ \downarrow \wr & & \downarrow \wr \\ \text{SFH}(S^3(L)) \otimes V & \longrightarrow & \text{SFH}(S^3(L)). \\ \mathbf{x} \otimes T & \longmapsto & 0 \\ \mathbf{x} \otimes B & \longmapsto & \mathbf{x}\end{array}$$

3.6.2. *The map associated to \mathcal{W}^b .* Recall that the cobordism \mathcal{W}^b is obtained by turning $\overline{\mathcal{W}}^b$ upside down. The cobordism $\overline{\mathcal{W}}^b$ is the disjoint union of the identity cobordism $\text{id}_{-S^3(L)}$ and a cobordism from \emptyset to (\mathbb{S}, γ) . By turning this disjoint union upside down,

$$\mathcal{W}^b = \text{id}_{S^3(L)} \sqcup \mathcal{W}',$$

where \mathcal{W}' is a cobordism from $-(\mathbb{S}, \gamma)$ to \emptyset . As \mathcal{W}' corresponds to an isolated component of Z , by [Juh09, Definition 10.1], we remove a standard contact ball from Z and add it to the outgoing end of \mathcal{W}' , making it a cobordism to $(B^3, \gamma_{\text{std}})$, when computing the map $F_{\mathcal{W}'}$.

By multiplicativity of the functor SFH , see [Juh09, Theorem 11.12, Axiom (4)],

$$F_{\mathcal{W}^b} = \text{id}_{\text{SFH}(S^3(L))} \otimes F_{\mathcal{W}'},$$

where

$$F_{\mathcal{W}'}: \text{SFH}(-(\mathbb{S}, \gamma)) \cong \mathbb{F}_2 \longrightarrow \text{SFH}(B^3, \gamma_{\text{std}}) \cong \mathbb{F}_2.$$

So $F_{\mathcal{W}'}$ is either zero or the unique isomorphism from \mathbb{F}_2 to \mathbb{F}_2 . As the death cobordism \mathcal{D} has a right inverse that is a split saddle cobordism, the map $F_{\mathcal{W}(\mathcal{D})}$ is surjective. But

$$F_{\mathcal{W}(\mathcal{D})} = F_{\mathcal{W}^b} \circ F_{\mathcal{W}^s} = (\text{id}_{\text{SFH}(S^3(L))} \otimes F_{\mathcal{W}'}) \circ F_{\mathcal{W}^s},$$

hence $F_{\mathcal{W}'}$ is non-zero. It follows that $F_{\mathcal{W}^b}$ is the composition of the isomorphisms $\text{SFH}(S^3(L) \sqcup -(\mathbb{S}, \gamma)) \cong \text{SFH}(S^3(L)) \otimes \text{SFH}(\mathbb{S}, \gamma) \cong \text{SFH}(S^3(L)) \otimes \mathbb{F}_2 \cong \text{SFH}(S^3(L))$.

By putting together the results of Sections 3.6.1 and 3.6.2, we obtain the map associated to a death cobordism:

Proposition 3.14. *Let \mathcal{D} be a death cobordism from $L \sqcup U_1$ to L . Then the following diagram is commutative:*

$$\begin{array}{ccc}
 \widehat{\text{HFL}}(L \sqcup U_1) & \xrightarrow{F_{\mathcal{D}}} & \widehat{\text{HFL}}(L) \\
 \downarrow \wr & & \downarrow \parallel \\
 \widehat{\text{HFL}}(L) \otimes V & \longrightarrow & \widehat{\text{HFL}}(L). \\
 x \otimes T & \longmapsto & 0 \\
 x \otimes B & \longmapsto & x
 \end{array}$$

Remark 3.15. The map associated to a birth cobordism and a death cobordism are dual to each other. If \mathcal{B} is a birth cobordism from L to $L \sqcup U_1$ (we omit the decorations for simplicity) and \mathcal{D} is the cobordism obtained by turning \mathcal{B} upside down, then the diagram

$$\begin{array}{ccc}
 \text{SFH}(S^3(L)) & \xrightarrow{F_{\mathcal{B}}} & \text{SFH}(S^3(L \sqcup U_1)) \\
 \downarrow \wr & & \downarrow \wr \\
 \text{SFH}(S^3(\overline{L}))^* & \xrightarrow{F_{\mathcal{D}}^*} & \text{SFH}(S^3(\overline{L \sqcup U_1}))^*,
 \end{array}$$

commutes, where the vertical isomorphisms are the ones given by duality; see the discussion in Section 3.6.1.

By replacing L by \overline{L} and dualizing, we obtain the following commutative diagram for a death cobordism \mathcal{D} from $L \sqcup U_1$ to L and \mathcal{B} the upside down cobordism:

$$\begin{array}{ccc}
 \text{SFH}(S^3(L \sqcup U_1)) & \xrightarrow{F_{\mathcal{D}}} & \text{SFH}(S^3(L)) \\
 \downarrow \wr & & \downarrow \wr \\
 \text{SFH}(S^3(\overline{L \sqcup U_1}))^* & \xrightarrow{F_{\mathcal{B}}^*} & \text{SFH}(S^3(\overline{L}))^*.
 \end{array}$$

3.7. Duality. We conclude this section with a few remarks about duality and gradings. If \mathcal{X} is a decorated link cobordism from (L_0, P_0) to (L_1, P_1) (or more generally, a sutured cobordism from (M_0, γ_0) to (M_1, γ_1)), then one can turn it upside down to get a decorated link cobordism from $(\overline{L}_1, \overline{P}_1)$ to $(\overline{L}_0, \overline{P}_0)$ (or more generally a sutured cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$).

In [Juh09, Theorem 11.9], the first author proved that if you turn a *special* cobordism upside down, then the map induced on SFH is the dual map, and asked [Juh09, Question 11.10] whether this is true for any sutured cobordism. A weaker question is whether this is true at least for decorated link cobordisms. An approach to answer this (weaker) question would be to check what happens for elementary cobordisms.

The positive (resp. negative) stabilisation and destabilisation are obtained from each other by turning the cobordism upside down, and we already checked that

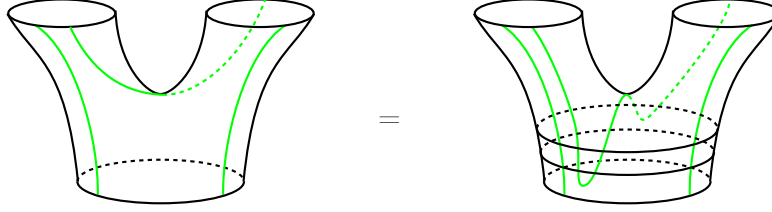


FIGURE 14. If we turn a saddle cobordism upside down, then we can write it as a composition of two stabilizations, followed by another saddle cobordism.

the maps that they induce are dual to each other. Analogously, the birth and the death cobordisms induce maps that are dual to each other as already observed in Remark 3.15.

The situation is more complicated for the saddle maps. First, note that if we turn a saddle cobordism upside down, then the height function has a local minimum on σ as opposed to a maximum, so this is not an elementary saddle cobordism. However, as in Figure 14, we can write this as a composition of two stabilizations and another saddle cobordism (which is a split if the original cobordism was a merge, and vice versa).

Let \mathcal{W} be the sutured cobordism associated to a saddle. As usual, we write it as $\mathcal{W}^s \circ \mathcal{W}^b$, where

- \mathcal{W}^b is a boundary cobordism from (M_0, γ_0) to (N, γ_1) , given by the gluing of a product annulus;
- \mathcal{W}^s is a special cobordism from (N, γ_1) to (M_1, γ_1) , consisting of a 2-handle attachment.

By turning this decomposition upside down, we obtain a decomposition for the upside down sutured cobordism $\overline{\mathcal{W}}$ as $\overline{\mathcal{W}}^b \circ \overline{\mathcal{W}}^s$, where

- $\overline{\mathcal{W}}^s$ is a special cobordism from $(-M_1, -\gamma_1)$ to $(-N, -\gamma_1)$ (consisting of the 2-handle dual to the one for \mathcal{W}^s);
- $\overline{\mathcal{W}}^b$ is a cobordism from $(-N, -\gamma_1)$ to $(-M_0, -\gamma_0)$ that is *not* a boundary cobordism.

The map $F_{\mathcal{W}^s}$ induced by the special cobordism is dual to the map $F_{\overline{\mathcal{W}}^s}$ by [Juh09, Theorem 11.9].

The map $F_{\overline{\mathcal{W}}^b}$ is not so clear. If one could prove that this is the inverse of the isomorphism induced by the product annulus decomposition

$$(-N, -\gamma_1) \rightsquigarrow (-M_0, -\gamma_0),$$

then the upside down cobordism would induce the dual map. Therefore, the question about duality in the decorated link category boils down to the following question.

Question 3.16. Let $(N, \gamma_1) \rightsquigarrow (M_0, \gamma_0)$ be a product annulus decomposition, and let \mathcal{W} be the boundary cobordism from (M_0, γ_0) to (N, γ_1) given by gluing along the product annulus. Does the map induced by $\overline{\mathcal{W}}$ coincide with the inverse of the inclusion induced by the product annulus decomposition?

	Alexander grading	Maslov grading
Positive (de)stabilisation	$+\frac{1}{2}$	$+\frac{1}{2}$
Negative (de)stabilisation	$-\frac{1}{2}$	$-\frac{1}{2}$
Birth or death	0	$+\frac{1}{2}$
Saddle	0	$-\frac{1}{2}$

TABLE 1. The table shows the shift in the Alexander grading and the Maslov grading associated to each elementary cobordism.

3.8. Gradings. Recall that $\widehat{\text{HFL}}$ of a link L in S^3 can be naturally endowed with two gradings, the Alexander grading (denoted by A) and the Maslov grading (denoted by M); cf. [OSz04, Ras03]. We will also consider the δ -grading, defined by $\delta = A - M$, following the convention established in [MO08].

The Alexander grading is defined on the chain level as

$$A(\mathbf{x}) = \frac{1}{2} \langle c_1(\mathfrak{s}(\mathbf{x})), [S] \rangle,$$

where S is any Seifert surface for the link and \mathbf{x} is a generator. Note that $\widehat{\text{HFL}}$ is a multi-graded theory, and collapsing the multi-grading onto the main diagonal gives the Alexander grading above. With only the Alexander grading, $\widehat{\text{HFL}}$ coincides with $\widehat{\text{HFK}}$, so we use them interchangeably. The Maslov grading is an absolute \mathbb{Q} -grading coming from the homological grading on the knot Floer complex. The main result of this section is the following theorem:

Theorem 3.17. *If $\mathcal{X} = (F, \sigma)$ is a decorated link cobordism, then the map induced on $\widehat{\text{HFL}}$ is homogeneous with respect to both the Alexander and the Maslov gradings. Furthermore, if \mathcal{X} is an elementary cobordism, then the shifts are as given in Table 1. Consequently, $F_{\mathcal{X}}$ shifts the Maslov grading by $\chi(F)/2 + (\chi(R_+(\sigma)) - \chi(R_-(\sigma)))/2$ and the Alexander grading by $(\chi(R_+(\sigma)) - \chi(R_-(\sigma)))/2$.*

Remark. As $\delta = A - M$, an immediate consequence of Theorem 3.17 is that $F_{\mathcal{X}}$ is also δ -homogeneous, and that the δ -grading shift is $-\chi(F)/2$ and therefore does not depend on the decorations.

Proof. Since every decorated cobordism is a product of elementary cobordisms by Proposition 3.6, it is sufficient to prove that the maps associated to elementary cobordisms are homogeneous.

Stabilisations are defined on the chain level as tensor products with a contact element, so they are homogeneous. The Alexander and the Maslov gradings of such an element (t or b) coincides with the grading shift. Destabilisation maps are dual to stabilisation maps, so they are also homogeneous, and the grading shifts are the same as for stabilisations.

The birth cobordism map is induced by the tensor product with the top-graded element of the vector space generated by two elements (called again T and B) in

different homological gradings, but in the same Alexander grading. Therefore the map is homogeneous and the grading shift is $(A, M) = (0, +\frac{1}{2})$. The death cobordism map is dual to the birth cobordism map, so the grading shift is the same.

By Theorem 3.11, the saddle map is given by an isomorphism induced by a product annulus – which does not change the gradings – followed by the map appearing in the decorated skein exact triangle. Ozsváth and Szabó proved in [OSz04, Theorem 10.2] that their map preserves the Alexander grading and drops the Maslov grading by $\frac{1}{2}$, so the saddle map does the same.

Finally, isotopies induce diffeomorphism maps that preserve both the Alexander and Maslov gradings. \square

4. ELEMENTARY LINK COBORDISMS BETWEEN UNLINKS

This section is devoted to computing the maps induced on HFL by elementary decorated link cobordisms between unlinks. In all our figures, the cobordisms go from the bottom to the top: If X is a cobordism from Y_0 to Y_1 , then Y_0 is on the bottom of the figure and Y_1 is on the top. Furthermore, we will denote by (U_n, P_{U_n}) , or, with slight abuse of notation, just by U_n , the standard n -component unlink with two decorations on each component. We will often consider it as embedded in the plane $\{z = 0\} \subseteq \mathbb{R}^3 \subseteq S^3$, with the centres of the components on the x axis, and for each component we will suppose that the x -coordinate of each point of $R_-(P_{U_n})$ (respectively, $R_+(P_{U_n})$) is less than or equal (respectively greater than or equal) to the x -coordinate of the centre; roughly speaking, we suppose that the $-$ arc is on the left and the $+$ arc is on the right. We will often omit the labels $-$ and $+$ in our figures. The figures depicting link cobordisms will follow the conventions explained in Figure 8: The cylinder enclosing a cobordism means that the cobordism is embedded in $S^3 \times I$ and that the figure represents the section $\{z = 0\}$ in $\mathbb{R}^3 \times I$, with framing as shown in Figure 8.

4.1. The split saddle cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) . Let $V = (S_V, \sigma_V)$ denote the split saddle cobordism from the unknot (U_1, P_{U_1}) to the 2-component unlink (U_2, P_{U_2}) , illustrated in Figure 15. We can break the sutured cobordism $\mathcal{W}(V)$ into a boundary cobordism \mathcal{W}^b and a special cobordism \mathcal{W}^s , as usual. By definition, the map F_V induced on HFL is the composition of two maps:

$$\text{HFL}(U_1) = \text{SFH}(S^3(U_1)) \xrightarrow{F_{\mathcal{W}^b}} \text{SFH}(N, \gamma_1) \xrightarrow{F_{\mathcal{W}^s}} \text{SFH}(S^3(U_2)) = \text{HFL}(U_2).$$

Note that $\text{SFH}(S^3(U_1)) \cong \mathbb{F}_2$ and $\text{SFH}(S^3(U_2)) \cong \mathbb{F}_2^2 \cong \mathbb{F}_2\langle B, T \rangle$ by Proposition 2.16, where B and T respectively denote the bottom-graded and the top-graded homogeneous generators of $\widehat{\text{HFL}}(U_2)$.

The cobordism V has a left inverse in the category **DLink**, which is the death cobordism L shown in Figure 15. Since $L \circ V = \text{id}_{U_1}$, by functoriality we know that $F_V : \mathbb{F}_2 \rightarrow \mathbb{F}_2^2$ is injective. Through a careful analysis of the special cobordism \mathcal{W}^s , we will prove that $\text{Im}(F_{\mathcal{W}^s}) \subseteq \mathbb{F}_2\langle B \rangle$, and therefore

$$\begin{aligned} F_V : \mathbb{F}_2 &\longrightarrow \mathbb{F}_2^2. \\ 1 &\longmapsto B \end{aligned} \tag{5}$$

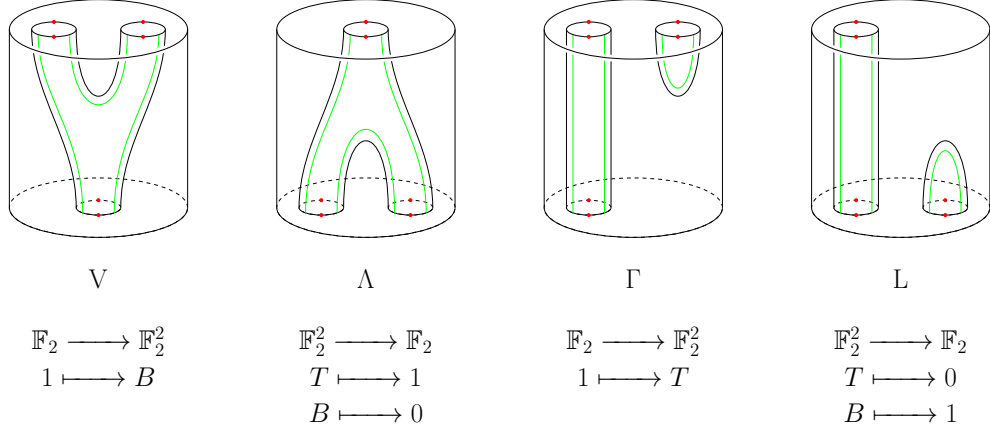


FIGURE 15. The decorated link cobordisms V , Λ , Γ , and L , and the maps induced on link Floer homology. As explained in Figure 8, the figures represent the section $\{z = 0\}$ in $\mathbb{R}^3 \times I$.

If we denote by $(M_0, \gamma_0) = S^3(U_1)$ and $(M_1, \gamma_1) = S^3(U_2)$, we have that

$$(M_0, \gamma_0) \cong (D^2 \times S^1, \{\pm 1\} \times S^1).$$

The surface S_V – a pair-of-pants shown in Figure 15 – is orientable, so its normal bundle in $S^3 \times I$ is trivial. It follows that the manifold N , which is the result of gluing M_0 and $-Z$, is diffeomorphic to $(S^1 \times I) \times S^1$. Thus, we can identify

$$(N, \gamma_1) = (S^1 \times I \times S^1, \{\pm 1\} \times \partial I \times S^1).$$

As explained in Section 3.4.2, the sutured manifold $S^3(U_2)$ is obtained from (N, γ_1) by attaching a 4-dimensional 2-handle: The attaching sphere is $S^1 \times \{0.5\} \times \{p\}$, with framing $S^1 \times \{0.5\} \times \{q\}$, where p and q are two points in S^1 . This handle is illustrated on the left-hand side of Figure 8. The map in SFH associated to such a 2-handle attachment is obtained by taking a sutured Heegaard diagram subordinate to a bouquet for the attaching sphere of the 2-handle. Such a sutured Heegaard diagram is shown in Figure 16. We will denote it by $(\Sigma, \alpha, \beta, \delta)$, where α , β , and δ denote the red, the blue, and the green curves, respectively. Let $M_{\varepsilon, \zeta}$ denote the sutured manifold defined by the sutured Heegaard diagram $(\Sigma, \varepsilon, \zeta)$. Then $M_{\alpha, \beta} \cong (N, \gamma_1)$ and $M_{\alpha, \delta} \cong S^3(U_2)$.

In the sutured Heegaard diagram (Σ, α, δ) , the sutured Floer complex has only two generators: the triple $[XY\Psi]$, which represents the bottom-graded generator B in homology, and the triple $[XY\Omega]$, which represents the top-graded generator T in homology. Call α_1 the α curve containing Ψ and Ω .

We say that a holomorphic triangle is *clockwise* if α , β , and δ appear along its sides opposite to the boundary orientation. The cobordism map is defined by counting clockwise holomorphic triangles of Maslov index 0 connecting a generator of $\text{CF}(\Sigma, \alpha, \beta)$, the top-graded generator of $\text{CF}(\Sigma, \beta, \delta)$, and a generator of $\text{CF}(\Sigma, \alpha, \delta)$. In the case at hand, one can check that there are no positive triangle domains with a corner at the generator $[XY\Omega]$ of $\text{CF}(\Sigma, \alpha, \delta)$: If there was such a domain

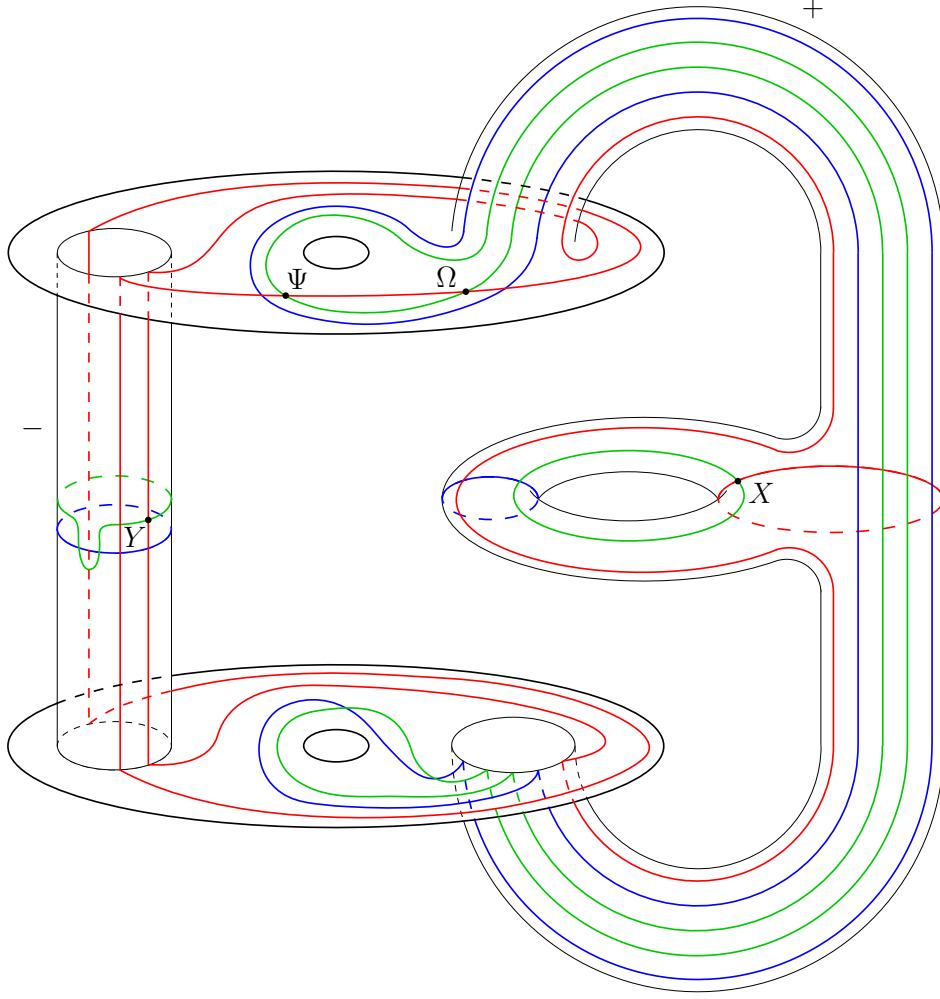


FIGURE 16. A sutured Heegaard diagram for the special cobordism \mathcal{W}^s associated to V . The α -curves are drawn in red, the β -curves in blue, and the δ -curves in green. The positive side of the surface Σ is the one with a $+$, and the negative one is denoted by a $-$.

$D \in \mathcal{D}(\Sigma, \alpha, \beta, \delta)$, then $\partial D \cap \alpha_1$ would be a 1-chain on α_1 with positive boundary at Ω , that is,

$$\partial(\partial D \cap \alpha_1) = \Omega - *.$$

However, this is impossible because the presence of the boundary components of Σ and the non-negativity of the coefficients of D imply that $\partial D \cap \alpha_1$ cannot have positive boundary at Ω . Indeed, the point of $\alpha \cap \beta$ closest to Ω does not appear in any element of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and D has multiplicity zero along $\partial\Sigma$, so $D \geq 0$ implies that $\partial D \cap \alpha_1$ points away from Ω (with non-negative multiplicity) on both sides of Ω .

As a consequence, the image of the chain map induced by the triple diagram $(\Sigma, \alpha, \beta, \delta)$ is entirely contained in $\mathbb{F}_2\langle[XY\Psi]\rangle$, and therefore the image of the map

induced on homology is contained in $\mathbb{F}_2\langle B \rangle$. As observed before, this uniquely determines the map F_V , which is the one in Equation (5).

Remark 4.1. Suppose that $\tilde{V} = (S_{\tilde{V}}, \sigma_{\tilde{V}})$ is a decorated cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) such that $S_{\tilde{V}} = S_V$, and such that there exists a left inverse to \tilde{V} . Note that the decorations $\sigma_{\tilde{V}}$ may be different from σ_V . Then, by applying the same argument as above, we can deduce that the map $F_{\tilde{V}}$ induced by \tilde{V} on SFH is again the one in Equation (5).

4.2. The merge saddle cobordism from (U_2, P_{U_2}) to (U_1, P_{U_1}) . We now turn our attention to the case of the merge saddle cobordism that we called Λ in Figure 15. Such a cobordism is obtained from V by reversing its orientation and turning it upside down. Recall that in Section 4.1 we decomposed the cobordism $\mathcal{W}(V)$ as a boundary cobordism \mathcal{W}^b composed with a special cobordism \mathcal{W}^s . By turning this decomposition upside down, we obtain a decomposition $\mathcal{W}(\Lambda) = \mathcal{W}(\Lambda)^b \circ \mathcal{W}(\Lambda)^s$, where $\mathcal{W}(\Lambda)^s$ is a special cobordism from $S^3(U_2)$ to the sutured manifold (N, γ_1) defined in Section 4.1 and $\mathcal{W}(\Lambda)^b$ is a (*not necessarily boundary*) cobordism from (N, γ_1) to $S^3(U_1)$. This implies that the map F_Λ is the composition of two maps:

$$\mathbb{F}_2^2 \cong \text{SFH}(S^3(U_2)) \xrightarrow{F_{\mathcal{W}(\Lambda)^s}} \text{SFH}(N, \gamma_1) \xrightarrow{F_{\mathcal{W}(\Lambda)^b}} \text{SFH}(S^3(U_1)) \cong \mathbb{F}_2.$$

Since Λ has a right inverse in the category **DLink**, which is the cobordism Γ illustrated in Figure 15, the map F_Λ is surjective. Therefore, in order to completely understand F_Λ , it suffices to determine its kernel. We now prove that B is in the kernel of the surgery map $F_{\mathcal{W}(\Lambda)^s}$, which implies that $\mathbb{F}_2\langle B \rangle = \ker(F_\Lambda)$, and therefore F_Λ is given by

$$\begin{aligned} F_\Lambda : \mathbb{F}_2^2 &\longrightarrow \mathbb{F}_2. \\ T &\longmapsto 1 \\ B &\longmapsto 0 \end{aligned} \tag{6}$$

The key observation, in order to prove that B is in the kernel of the surgery map $F_{\mathcal{W}(\Lambda)^s}$, is that the special cobordism $\mathcal{W}(\Lambda)^s$ from $S^3(U_2)$ to (N, γ_1) consists of the attachment of a 2-handle which is dual to the one on the left-hand side of Figure 8. It follows that it can be expressed by the sutured Heegaard diagram obtained from the one in Figure 16 by swapping the β - and the δ -curves. Thus, the special cobordism map $F_{\mathcal{W}(\Lambda)^s}$ can be computed by counting *counterclockwise* triangles of Maslov index 0 in the sutured Heegaard diagram in Figure 16.

If there was a counterclockwise domain $D \in \mathcal{D}(\Sigma, \alpha, \delta, \beta)$ with non-negative multiplicities that has a corner at the generator $[XY\Psi]$, then we could deduce that $\partial D \cap \alpha_1$ has a negative boundary point at Ψ ; i.e.,

$$\partial(\partial D \cap \alpha_1) = * - \Psi.$$

However, the presence of the boundary components of Σ rules out this possibility. Therefore $[XY\Psi]$ lies in the kernel of the chain map. This implies that $F_{\mathcal{W}(\Lambda)^s}(B) = 0$, and, from the discussion above, that the map F_Λ is given by Equation (6).

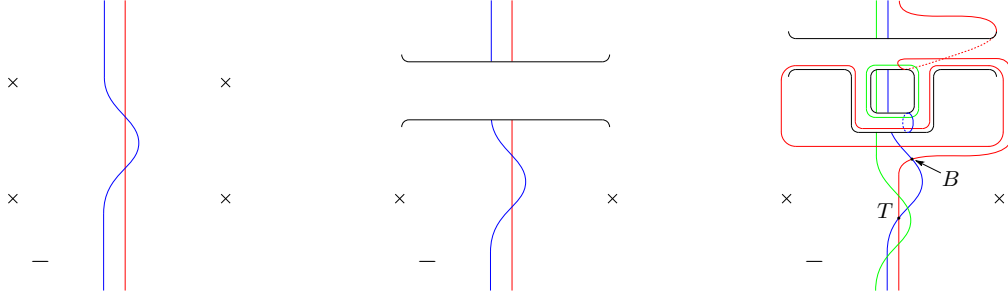


FIGURE 17. The figure on the left shows a Heegaard diagram for (U_2, P_{U_2}) , where the Heegaard surface is S^2 . The figures in the middle and on the right show the two steps of the Heegaard diagrammatic description of the map F_Λ as from Section 3.4.

Remark 4.2. Suppose that $\tilde{\Lambda} = (S_{\tilde{\Lambda}}, \sigma_{\tilde{\Lambda}})$ is a cobordism from (U_2, P_{U_2}) to (U_1, P_{U_1}) such that $S_{\tilde{\Lambda}} = S_\Lambda$ and such that $\tilde{\Lambda}$ has a right inverse. Note that the decorations $\sigma_{\tilde{\Lambda}}$ may be different from σ_Λ . Then, by applying the same argument as above, we can deduce that the map $F_{\tilde{\Lambda}}$ induced by $\tilde{\Lambda}$ in SFH is again the one in Equation (6).

Remark. One can also use the Heegaard diagrammatic approach described in Section 3.4 to prove that the map induced by the cobordism Λ is the one in Equation (6). Start from the Heegaard diagram for (U_2, P_{U_2}) as on the left-hand side of Figure 17. This is not a Heegaard diagram arising from a connected link projection of (U_2, P_{U_2}) , but the arguments in Section 3.4 work without modifications. Therefore, the middle part of Figure 17 shows a Heegaard diagram for the sutured manifold after the gluing map, and the right-hand side of Figure 17 shows a Heegaard diagram for the 2-handle attachment which the special cobordism consists of. If now D is the domain of a Whitney triangle with some holomorphic representative connecting the bottom-graded generator $B \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to some generators in $\text{CF}(\beta, \delta)$ and in $\text{CF}(\alpha, \delta)$, then $\partial D \cap \alpha$ should emanate from B . This is impossible because, due to the presence of the sutures, $\partial D \cap \alpha$ can only point into B . It follows that $B \in \ker(F_\Lambda)$, which in turn implies Equation (6).

4.3. The birth cobordism Γ and the death cobordism L . Consider the birth cobordism Γ from (U_1, P_{U_1}) to (U_2, P_{U_2}) shown in Figure 15, and let L denote the upside down death cobordism. It follows from Sections 3.5 and 3.6 that the maps induced by these cobordisms are

$$\begin{aligned} F_\Gamma : \mathbb{F}_2 &\longrightarrow \mathbb{F}_2^2 \\ 1 &\longmapsto T \end{aligned} \tag{7}$$

and

$$\begin{aligned} F_L : \mathbb{F}_2^2 &\longrightarrow \mathbb{F}_2. \\ T &\longmapsto 0 \\ B &\longmapsto 1 \end{aligned} \tag{8}$$

Remark. All the computations in Section 4 are summarised in Figure 15.

5. A DISJOINT UNION FORMULA

In this section, we focus on a formula for the disjoint union of link cobordisms, with the aim of putting the maps associated to the cobordisms in Figure 15 into a TQFT. Although we actually need a minimal version of it, we give a proof in the general setting that might be of independent interest; cf. Theorem 5.9.

It is well known (cf. for instance [Juh06, Proposition 9.15]) that the link Floer homology of the “split” disjoint union of two links K and L satisfies

$$(9) \quad \text{HFL}(K \sqcup L) \cong \text{HFL}(K) \otimes \text{HFL}(L) \otimes \mathbb{F}_2^2.$$

We would like to understand if the cobordism map associated to a disjoint union of link cobordisms splits in a similar way.

Definition 5.1. Suppose that $\mathcal{X}_K = (F, \sigma_F)$ is a decorated link cobordism from (K_0, P_{K_0}) to (K_1, P_{K_1}) , and that $\mathcal{X}_L = (G, \sigma_G)$ is a decorated link cobordism from (L_0, P_{L_0}) to (L_1, P_{L_1}) . We define the *disjoint union cobordism* to be the cobordism in $S^3 \times I$ obtained by taking the split disjoint union of the two surfaces F and G with decoration $\sigma_F \sqcup \sigma_G$. We denote it by $\mathcal{X}_K \# \mathcal{X}_L$. Such a cobordism connects the decorated link $(K_0 \sqcup L_0, P_{K_0} \sqcup P_{L_0})$ (the “split” disjoint union of (K_0, P_{K_0}) and (L_0, P_{L_0})) to the decorated link $(K_1 \sqcup L_1, P_{K_1} \sqcup P_{L_1})$.

One would hope that the map induced by the cobordism $\mathcal{X}_K \# \mathcal{X}_L$ on sutured Floer homology is the tensor product of the maps induced respectively by \mathcal{X}_K and \mathcal{X}_L and the map $\text{id}_{\mathbb{F}_2^2}$. We will show that this is true under some mild restrictions.

We will actually prove it in a generalised setting, by considering sutured manifolds. Recall that the complement of a decorated link is naturally a sutured manifold; cf. Definition 2.11. By [Juh06, Proposition 9.15], Equation (9) generalises as follows: If (M, γ) and (N, ν) are balanced sutured manifolds, then

$$(10) \quad \text{SFH}((M, \gamma) \# (N, \nu)) \cong \text{SFH}(M, \gamma) \otimes \text{SFH}(N, \nu) \otimes \mathbb{F}_2^2.$$

Actually, we need to make the splitting in Equation (10) functorial, and for this purpose we use *framed pairs of points*, *bouquets*, and *sutured diagrams subordinate to bouquets* as defined in [Juh09, Section 7].

Given a framed pair of points \mathbb{P} for $(M, \gamma) \sqcup (N, \nu)$ such that $p_- \in M$ and $p_+ \in N$, let $M^\circ = \text{Bl}_{p_-}(M)$ and $N^\circ = \text{Bl}_{p_+}(N)$. We can view M° and N° as embedded in $M \#_{\mathbb{P}} N$. Denote by S^2 the sphere $M^\circ \cap N^\circ$. The relative Mayer-Vietoris sequences in cohomology for $(M, \partial M) = (M^\circ, \partial M) \cup B^3$, $(N, \partial N) = (N^\circ, \partial N) \cup B^3$, and $(M \#_{\mathbb{P}} N, \partial(M \#_{\mathbb{P}} N)) = (M^\circ, \partial M) \cup (N^\circ, \partial N)$ imply that the map

$$\begin{aligned} \phi : \text{Spin}^c(M, \gamma) \times \text{Spin}^c(N, \nu) &\longrightarrow \text{Spin}^c(M \#_{\mathbb{P}} N, \gamma \cup \nu) \\ (\mathfrak{s}_M, \mathfrak{s}_N) &\longmapsto (\mathfrak{s}_M)|_{M^\circ} \cup (\mathfrak{s}_N)|_{N^\circ} \end{aligned} \quad (11)$$

is well-defined and injective. We will write $\mathfrak{s}_M \#_{\mathbb{P}} \mathfrak{s}_N = \phi(\mathfrak{s}_M, \mathfrak{s}_N)$. Furthermore, $\mathfrak{s} \in \text{Spin}^c(M \#_{\mathbb{P}} N, \gamma \cup \nu)$ is in the image of ϕ if and only if $\mathfrak{s}|_{S^2} = 0$ in $\text{Spin}^c(S^2)$; i.e., if and only if it is the unique Spin^c structure on S^2 with vanishing first Chern class. In this case, we call $\mathfrak{s}|_M$ and $\mathfrak{s}|_N$ the unique Spin^c structures such that $\phi(\mathfrak{s}|_M, \mathfrak{s}|_N) = \mathfrak{s}$.

The following is a Spin^c refinement of [Juh06, Proposition 9.15].

Lemma 5.2. *Let \mathbb{P} be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. Let $\mathcal{H}_M = (\Sigma_M, \alpha_M, \beta_M)$ and $\mathcal{H}_N = (\Sigma_N, \alpha_N, \beta_N)$ be admissible balanced diagrams such that $\mathcal{H}_M \sqcup \mathcal{H}_N$ is subordinate to a bouquet $B(\mathbb{P})$ for \mathbb{P} . If $\mathfrak{s} \in \text{Spin}^c(M \#_{\mathbb{P}} N, \gamma \sqcup \nu)$ is such that $\mathfrak{s}|_{S^2} \neq 0$, then*

$$\text{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) = 0.$$

If $\mathfrak{s}|_{S^2} = 0$, then there is an isomorphism

$$\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}, \mathfrak{s}}: \text{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) \rightarrow \text{SFH}(\mathcal{H}_M, \mathfrak{s}|_M) \otimes \text{SFH}(\mathcal{H}_N, \mathfrak{s}|_N) \otimes V,$$

where $V = \mathbb{F}_2\langle T, B \rangle$ denotes a 2-dimensional vector space generated by two homogeneous elements T (top) and B (bottom). Moreover, there is an isomorphism

$$\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}}: \text{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N) \rightarrow \text{SFH}(\mathcal{H}_M) \otimes \text{SFH}(\mathcal{H}_N) \otimes V.$$

Proof. Notice that for each generator $\mathbf{x} \in \text{CF}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N)$, we have $\mathfrak{s}(\mathbf{x})|_{S^2} = 0$. Therefore, $\text{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) = 0$ unless $\mathfrak{s}|_{S^2} = 0$. There is a self-evident bijection

$$\text{CF}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) \rightarrow \text{CF}(\mathcal{H}_M, \mathfrak{s}|_M) \otimes \text{CF}(\mathcal{H}_N, \mathfrak{s}|_N) \otimes V.$$

Since $\mathcal{H}_M \sqcup \mathcal{H}_N$ is subordinate to the bouquet $B(\mathbb{P})$, all holomorphic discs on $\Sigma_M \# \Sigma_N$ split as the disjoint union of a holomorphic disc on Σ_M , one on Σ_N , and one on the annulus A attached to the framed pair of points \mathbb{P} . Each of these discs has multiplicity 0 near the boundary. It follows that the above bijection descends to a map on homology. \square

The following Lemma proves that the isomorphisms defined in Lemma 5.2 are natural.

Lemma 5.3. *Let \mathbb{P} be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. Let $\mathcal{H}_M = (\Sigma_M, \alpha_M, \beta_M)$ and $\mathcal{H}'_M = (\Sigma'_M, \alpha'_M, \beta'_M)$ be admissible balanced diagrams for (M, γ) and $\mathcal{H}_N = (\Sigma_N, \alpha_N, \beta_N)$ and $\mathcal{H}'_N = (\Sigma'_N, \alpha'_N, \beta'_N)$ be admissible balanced diagrams for (N, ν) such that $\mathcal{H}_M \sqcup \mathcal{H}_N$ and $\mathcal{H}'_M \sqcup \mathcal{H}'_N$ are subordinate to the bouquets $B(\mathbb{P})$ and $B(\mathbb{P})'$ for \mathbb{P} , respectively. Let $\mathfrak{s}_M \in \text{Spin}^c(M, \gamma)$ and $\mathfrak{s}_N \in \text{Spin}^c(N, \nu)$, and let $\mathfrak{s}_{\mathbb{P}} = \mathfrak{s}_M \#_{\mathbb{P}} \mathfrak{s}_N \in \text{Spin}^c(M \#_{\mathbb{P}} N, \gamma \sqcup \nu)$ denote the connected sum of the Spin^c structures. Then, if $\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N$ and $\mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N$ denote the Heegaard diagrams obtained from $\mathcal{H}_M \sqcup \mathcal{H}_N$ and $\mathcal{H}'_M \sqcup \mathcal{H}'_N$ by surgery on \mathbb{P} , we have that the following diagram commutes:*

$$\begin{array}{ccc} \text{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}_{\mathbb{P}}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}, \mathfrak{s}_{\mathbb{P}}}} & \text{SFH}(\mathcal{H}_M, \mathfrak{s}_M) \otimes \text{SFH}(\mathcal{H}_N, \mathfrak{s}_N) \otimes V \\ \downarrow F_{\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N} & & \downarrow F_{\mathcal{H}_M, \mathcal{H}'_M} \otimes F_{\mathcal{H}_N, \mathcal{H}'_N} \otimes \text{id}_V \\ \text{SFH}(\mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N, \mathfrak{s}_{\mathbb{P}}) & \xrightarrow{\varphi_{\mathcal{H}'_M, \mathcal{H}'_N, \mathbb{P}, \mathfrak{s}_{\mathbb{P}}}} & \text{SFH}(\mathcal{H}'_M, \mathfrak{s}_M) \otimes \text{SFH}(\mathcal{H}'_N, \mathfrak{s}_N) \otimes V. \end{array}$$

Here $F_{\mathcal{H}, \mathcal{H}'}$ denotes the naturality map defined in [JT12]. An analogous statement holds for $\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}}$ and $\varphi_{\mathcal{H}'_M, \mathcal{H}'_N, \mathbb{P}}$.

Proof. The proof is analogous to [Juh09, Theorem 7.6]. \square

Corollary 5.4. *Let \mathbb{P} be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. If $\mathfrak{s} \in \text{Spin}^c(M \#_{\mathbb{P}} N, \gamma \sqcup \nu)$ is such that $\mathfrak{s}|_{S^2} \neq 0$, then*

$$\text{SFH}(M \#_{\mathbb{P}} N, \mathfrak{s}) = 0.$$

If $\mathfrak{s}|_{S^2} = 0$, then there is a well-defined isomorphism

$$\varphi_{\mathbb{P}, \mathfrak{s}}: \text{SFH}(M \#_{\mathbb{P}} N, \mathfrak{s}) \rightarrow \text{SFH}(M, \mathfrak{s}|_M) \otimes \text{SFH}(N, \mathfrak{s}|_N) \otimes V.$$

Moreover, there is an isomorphism

$$\varphi_{\mathbb{P}}: \text{SFH}(M \#_{\mathbb{P}} N) \rightarrow \text{SFH}(M) \otimes \text{SFH}(N) \otimes V.$$

We now generalise the notion of framed pair of points to cobordisms, in order to define the connected sum of sutured cobordisms in a natural way.

Definition 5.5. Let $\mathcal{W} = (W, Z_W, [\xi_W])$ (resp. $\mathcal{U} = (U, Z_U, [\xi_U])$) be a cobordism between sutured manifolds (M_0, γ_0) and (M_1, γ_1) (resp. (N_0, ν_0) and (N_1, ν_1)). A *framed pair of arcs* \mathbb{A} consists of two properly embedded arcs $a_-: I \rightarrow W$ and $a_+: I \rightarrow U$, such that $a_-(i) \in M_i$ and $a_+(i) \in N_i$ for $i \in \{0, 1\}$, together with a negative frame $\langle v_1^-(t), v_2^-(t), v_3^-(t), v_4^-(t) \rangle$ of $T_{a_-(t)}W$ and a positive frame $\langle v_1^+(t), v_2^+(t), v_3^+(t), v_4^+(t) \rangle$ of $T_{a_+(t)}U$ for all $t \in I$ such that:

- for $i \in \{0, 1\}$, $\mathbb{A}(i) := \{a_-(i), a_+(i)\}$ with the frames $\langle v_1^\pm(i), v_2^\pm(i), v_3^\pm(i) \rangle$ is a framed pair of points for $(M_i \sqcup N_i, \gamma_i \sqcup \nu_i)$;
- $v_4 = a'_\pm = da_\pm/dt$.

Given a framed pair of arcs, we can define the connected sum of the two cobordisms, as explained in Definition 5.6, below. We denote by $(M, \gamma) \#_{\mathbb{P}} (N, \nu)$ the connected sum of the sutured manifolds (M, γ) and (N, ν) along the framed pair of points \mathbb{P} , assuming that $p_- \in M$ and $p_+ \in N$.

Definition 5.6. Let $\mathcal{W} = (W, Z_W, [\xi_W])$ (resp. $\mathcal{U} = (U, Z_U, [\xi_U])$) be a cobordism between sutured manifolds (M_0, γ_0) and (M_1, γ_1) (resp. (N_0, ν_0) and (N_1, ν_1)), and let \mathbb{A} be a framed pair of arcs.

We define the *connected sum* of \mathcal{W} and \mathcal{U} as the cobordism $\mathcal{W} \#_{\mathbb{A}} \mathcal{U} = (V, Z_V, [\xi_V])$ from $(M_0, \gamma_0) \#_{\mathbb{A}(0)} (N_0, \nu_0)$ to $(M_1, \gamma_1) \#_{\mathbb{A}(1)} (N_1, \nu_1)$, where

- V is the 4-dimensional manifold obtained by gluing $\text{Bl}_{a_-} W$ and $\text{Bl}_{a_+} U$ respectively along UNa_- and UNa_+ , in such a way that the frame of $a_-(t)$ is identified with the one of $a_+(t)$;
- $Z_V = Z_W \sqcup Z_U$;
- $\xi_V = \xi_W \sqcup \xi_U$.

Remark 5.7. Let $\mathcal{X}_K = (F, \sigma_F)$ (resp. $\mathcal{X}_L = (G, \sigma_G)$) be a decorated link cobordism from (K_0, P_{K_0}) to (K_1, P_{K_1}) (resp. from (L_0, P_{L_0}) to (L_1, P_{L_1})). Let $(x_-, y_-, z_-) \in S^3$ be such that

$$a_- = \{ (x_-, y_-, z_-, t) : t \in I \}$$

is disjoint from F . This holds for generic (x_-, y_-, z_-) , after possibly isotoping F . Analogously, choose $(x_+, y_+, z_+) \in S^3$ and define a_+ to be disjoint from G . Consider the pair of arcs $\mathbb{A} = \{a_-, a_+\}$ with frames $\langle \pm \partial_x, \partial_y, \partial_z, \partial_t \rangle$. Then

$$\mathcal{W}(\mathcal{X}_K \#_{\mathbb{A}} \mathcal{X}_L) = \mathcal{W}(\mathcal{X}_K) \#_{\mathbb{A}} \mathcal{W}(\mathcal{X}_L).$$

As in the case of the connected sum of sutured manifolds, one can define the connected sum of Spin^c structures for cobordisms. Let $\mathcal{W} = (W, Z_W, [\xi_W])$ and $\mathcal{U} = (U, Z_U, [\xi_U])$ be cobordisms of sutured manifolds, and let $\mathcal{W} \#_{\mathbb{A}} \mathcal{U} = (W \#_{\mathbb{A}} U, Z, [\xi])$ denote their connected sum along some framed pair of arcs \mathbb{A} . Let $W^\circ = \text{Bl}_{a_-} W$ and $U^\circ = \text{Bl}_{a_+} U$. Then W° and U° can be seen as embedded in $W \#_{\mathbb{A}} U$. Denote by $S^2 \times I$ the thickened sphere obtained by intersecting $W^\circ \cap U^\circ$. Then, the relative Mayer-Vietoris sequences for $(W, Z_W) = (W^\circ, Z_W) \cup B^3 \times I$, $(U, Z_U) = (U^\circ, Z_U) \cup B^3 \times I$, and $(W \#_{\mathbb{A}} U, Z) = (W^\circ, Z_W) \cup (U^\circ, Z_U)$ imply that the map

$$\begin{aligned} \phi : \text{Spin}^c(W, Z_W) \times \text{Spin}^c(U, Z_U) &\longrightarrow \text{Spin}^c(W \#_{\mathbb{A}} U, Z) \\ (\mathfrak{s}_W, \mathfrak{s}_U) &\longmapsto (\mathfrak{s}_W)|_{W^\circ} \cup (\mathfrak{s}_U)|_{U^\circ} \end{aligned}$$

is well-defined and injective. We will denote $\phi(\mathfrak{s}_W, \mathfrak{s}_U)$ by $\mathfrak{s}_W \#_{\mathbb{A}} \mathfrak{s}_U$. Moreover, $\mathfrak{s} \in \text{Spin}^c(W \#_{\mathbb{A}} U, Z)$ is in the image of ϕ if and only if $\mathfrak{s}|_{S^2 \times I} = 0$ in $\text{Spin}^c(S^2 \times I)$ (i.e., if and only if it is the unique Spin^c structure on $S^2 \times I$ with vanishing first Chern class). In such a case we call $\mathfrak{s}|_W$ and $\mathfrak{s}|_U$ the unique Spin^c structures such that $\phi(\mathfrak{s}|_W, \mathfrak{s}|_U) = \mathfrak{s}$.

Definition 5.8. Let \mathbb{A} be a framed pair of arcs for $\mathcal{W} = (W, Z_W, [\xi_W])$ and $\mathcal{U} = (U, Z_U, [\xi_U])$. A *bouquet* $B(\mathbb{A})$ for \mathbb{A} consists of a pair of embedded squares $d_- : I \times I \rightarrow W$ and $d_+ : I \times I \rightarrow U$ and a framing of d_\pm given by a normal vector field v_\pm such that:

- $d_\pm(0, t) = a_\pm(t)$ and $d_\pm(1, t) \in Z_W \cup Z_U$ for all $t \in I$;
- for $i \in \{0, 1\}$, $d_\pm(s, i)$ with $v_\pm(s, i)$, which we denote by $B(\mathbb{A})_i$, is a bouquet for $\mathbb{A}(i)$ (in particular, $d_-(s, i)$ lies in M_i and $d_+(s, i)$ lies in N_i);
- $\partial_s d_\pm(0, t) = v_1^\pm$ and $\partial_s d_\pm(1, t)$ is transverse to Z ;
- $v_\pm(0, t) = v_2^\pm$;
- along $d_\pm(1, t)$, $\xi = \langle \partial_s d_\pm, v_\pm \rangle$;
- $\partial_t d_\pm(0, t) = v_4^\pm$ and $\partial_t d_\pm(s, i)$ is transverse to $M_i \sqcup N_i$ for $i \in \{0, 1\}$.

Theorem 5.9. Let $\mathcal{W} = (W, Z_W, [\xi_W])$ (respectively $\mathcal{U} = (U, Z_U, [\xi_U])$) be a cobordism between the balanced sutured manifolds (M_0, γ_0) and (M_1, γ_1) (respectively (N_0, ν_0) and (N_1, ν_1)). Let \mathbb{A} be a framed pair of arcs for the cobordisms \mathcal{W} and \mathcal{U} , and suppose that there exists some bouquet $B(\mathbb{A})$ for \mathbb{A} . Then the following diagram commutes (the sutures have been dropped from the notations because they are clear).

$$\begin{array}{ccc} \text{SFH}(M_0 \#_{\mathbb{A}(0)} N_0) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \text{SFH}(M_0) \otimes \text{SFH}(N_0) \otimes V \\ \downarrow F_{\mathcal{W} \#_{\mathbb{A}} \mathcal{U}} & & \downarrow F_{\mathcal{W}} \otimes F_{\mathcal{U}} \otimes \text{id}_V \\ \text{SFH}(M_1 \#_{\mathbb{A}(1)} N_1) & \xrightarrow{\varphi_{\mathbb{A}(1)}} & \text{SFH}(M_1) \otimes \text{SFH}(N_1) \otimes V \end{array}$$

Example 5.10. The existence of a bouquet $B(\mathbb{A})$ in the statement of Theorem 5.9 is not just a technical condition, as we are going to show in this example. Consider the cobordism $\Gamma \circ \text{L}$ (cf. Figure 15 for a definition of the cobordisms Γ and L). By the

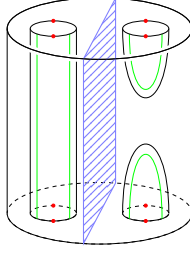


FIGURE 18. The cobordism $\Gamma \circ L$ can be seen as the disjoint union cobordism of two cobordisms from (U_1, P_{U_1}) to (U_1, P_{U_1}) , namely the identity cobordism to the left of the blue rectangle, which we denote by \mathcal{X}_1 , and the one to its right, which we denote by \mathcal{X}_2 .

computations in Section 4 and functoriality, we know that the map

$$F_{\Gamma \circ L} : \mathbb{F}_2^2 \longrightarrow \mathbb{F}_2^2$$

induced on SFH has rank 1.

This cobordism can also be seen as the disjoint union $\mathcal{X}_1 \amalg \mathcal{X}_2$ of the identity cobordism \mathcal{X}_1 from (U_1, P_{U_1}) to (U_1, P_{U_1}) and a “death and re-birth” cobordism \mathcal{X}_2 again from (U_1, P_{U_1}) to (U_1, P_{U_1}) ; cf. Figure 18. Choose \mathbb{A} as in Remark 5.7. Then we have that

$$F_{\Gamma \circ L} = F_{\mathcal{W}(\mathcal{X}_1) \amalg_{\mathbb{A}} \mathcal{W}(\mathcal{X}_2)}.$$

We cannot apply Theorem 5.9 here because there is not a bouquet for \mathbb{A} . If the diagram in Theorem 5.9 were commutative also in this case, then the map induced by $\mathcal{W}(\mathcal{X}_1) \amalg_{\mathbb{A}} \mathcal{W}(\mathcal{X}_2)$ would have rank either 0 or 2, contradicting our computations.

Proof of Theorem 5.9. Split \mathcal{W} and \mathcal{U} into a boundary cobordism and a special cobordism: $\mathcal{W} = \mathcal{W}^s \circ \mathcal{W}^b$ and $\mathcal{U} = \mathcal{U}^s \circ \mathcal{U}^b$. Then \mathbb{A} determines framed pairs of arcs \mathbb{A}^b for \mathcal{W}^b and \mathcal{U}^b and \mathbb{A}^s for \mathcal{W}^s and \mathcal{U}^s . Note that

$$\mathcal{W} \amalg_{\mathbb{A}} \mathcal{U} = (\mathcal{W}^s \amalg_{\mathbb{A}^s} \mathcal{U}^s) \circ (\mathcal{W}^b \amalg_{\mathbb{A}^b} \mathcal{U}^b)$$

is a decomposition of $\mathcal{W} \amalg_{\mathbb{A}} \mathcal{U}$ into a boundary cobordism and a special cobordism.

Fix a bouquet $B(\mathbb{A})$ for \mathbb{A} . Note that this induces bouquets $B(\mathbb{A}^b)$ and $B(\mathbb{A}^s)$. Therefore, we can consider the cases of boundary cobordism and special cobordism separately. We now split the proof into three cases.

Case 1: \mathcal{W} and \mathcal{U} are identity cobordisms. We first consider the case when \mathcal{W} and \mathcal{U} are identity cobordisms, and \mathbb{A} is any framed pair of arcs that admits a bouquet $B(\mathbb{A})$. Let B_- be a regular neighbourhood of $\text{Im}(d_-)$ in W , and let B_+ be a regular neighbourhood of $\text{Im}(d_+)$ in U . For $i \in \{0, 1\}$, let $B_{\pm}(i) = B_{\pm} \cap (M_i \sqcup N_i)$. The fact that the contact structure ξ is parallel near the bouquet implies that B_{\pm} can be turned into a special cobordism

$$\mathcal{B}_{\pm} : B_{\pm}(0) \rightarrow B_{\pm}(1)$$

by extending ξ to $\partial B_{\pm} \cap \text{Int}(W \cup U)$. If we remove \mathcal{B}_{\pm} from \mathcal{W} and \mathcal{U} , we obtain two identity cobordisms $\mathcal{W}^{\circ} = \text{id}_M$ and $\mathcal{U}^{\circ} = \text{id}_N$, where $M = M_0 \setminus B_-(0)$ and $N = N_0 \setminus B_+(0)$. Note that \mathbb{A} is a framed pair of arcs and $B(\mathbb{A})$ is a bouquet in \mathcal{B}_{\pm} .

The cobordism $\mathcal{B}_- \#_{\mathbb{A}} \mathcal{B}_+$ is diffeomorphic to an identity cobordism, therefore it gives a diffeomorphism

$$\psi: B_-(0) \#_{\mathbb{A}(0)} B_+(0) \rightarrow B_-(1) \#_{\mathbb{A}(1)} B_+(1),$$

and, without loss of generality, we can suppose that

- $\psi(a_{\pm}(s, 0)) = a_{\pm}(s, 1)$, and
- $d\psi(v_{\pm}(s, 0)) = v_{\pm}(s, 1)$.

Let $D_{\pm}(0)$ be a 2-disc embedded in $B_{\pm}(0)$ containing $a_{\pm}(\cdot, 0)$ such that $v_{\pm}(\cdot, 0)$ is tangent to $D_{\pm}(0)$, and let $D_{\pm}(1) = \psi(D_{\pm}(0))$. Then $D_-(i) \#_{\mathbb{A}(i)} D_+(i)$, together with a meridional α - and a β -curve on the connecting tube that intersect in two points, is a Heegaard diagram \mathcal{H}_i for $B_-(i) \#_{\mathbb{A}(i)} B_+(i)$, and $\psi_*(\mathcal{H}_0) = \mathcal{H}_1$. The map induced by such a diffeomorphism on sutured Floer homology preserves the relative homological grading, so it must be the identity map:

$$\begin{array}{ccc} \mathrm{SFH}(\mathcal{H}_0) & \xrightarrow{\cong} & V \\ \downarrow \psi_* & & \downarrow \mathrm{id}_V \\ \mathrm{SFH}(\mathcal{H}_1) & \xrightarrow{\cong} & V. \end{array}$$

We then choose Heegaard diagrams \mathcal{H}_M and \mathcal{H}_N for the sutured manifolds M and N such that

$$\mathcal{H}_{M \#_{\mathbb{A}(0)} N} = \mathcal{H}_M \natural \mathcal{H}_0 \natural \mathcal{H}_N$$

is a Heegaard diagram for the sutured manifold $M \#_{\mathbb{A}(0)} N$ (or, more precisely, for $M_0 \#_{\mathbb{A}(0)} N_0$). The cobordism $\mathcal{W} \#_{\mathbb{A}} \mathcal{U}$ induces a diffeomorphism

$$\Psi_*: \mathcal{H}_{M \#_{\mathbb{A}(0)} N} \rightarrow \mathcal{H}_{M \#_{\mathbb{A}(1)} N} = \mathcal{H}_M \natural \mathcal{H}_1 \natural \mathcal{H}_N$$

such that the following diagram of diffeomorphisms commutes:

$$\begin{array}{ccc} \mathcal{H}_{M \#_{\mathbb{A}(0)} N} & \xrightarrow{\cong} & \mathcal{H}_M \natural \mathcal{H}_0 \natural \mathcal{H}_N \\ \downarrow \Psi_* & & \downarrow \mathrm{id}_{\mathcal{H}_M} \natural \psi_* \natural \mathrm{id}_{\mathcal{H}_N} \\ \mathcal{H}_{M \#_{\mathbb{A}(1)} N} & \xrightarrow{\cong} & \mathcal{H}_M \natural \mathcal{H}_1 \natural \mathcal{H}_N. \end{array}$$

From this, it follows that the diagram below commutes:

$$\begin{array}{ccc} \mathrm{SFH}(\mathcal{H}_{M \#_{\mathbb{A}(0)} N}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_M) \otimes \mathrm{SFH}(\mathcal{H}_N) \otimes V \\ \downarrow F_{\mathcal{H}_{M \#_{\mathbb{A}(0)} N}, \mathcal{H}_{M \#_{\mathbb{A}(1)} N}} & & \downarrow \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_M)} \otimes \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_N)} \otimes \mathrm{id}_V \\ \mathrm{SFH}(\mathcal{H}_{M \#_{\mathbb{A}(1)} N}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{A}(1)}} & \mathrm{SFH}(\mathcal{H}_M) \otimes \mathrm{SFH}(\mathcal{H}_N) \otimes V. \end{array}$$

Case 2: \mathcal{W} and \mathcal{U} are boundary cobordisms. Since \mathcal{W} and \mathcal{U} are boundary cobordisms, recall that $M_1 \cong M_0 \cup -Z_W$ and $N_1 \cong N_0 \cup -Z_U$. Given a bouquet $B(\mathbb{A}) = \{d_\pm, v_\pm\}$ for \mathbb{A} , we define a bouquet $B'(\mathbb{A}(0))$ for $\mathbb{A}(0)$ in the sutured manifold $(M_1, \gamma_1) \sqcup (N_1, \nu_1)$. The arcs of $B'(\mathbb{A}(0))$ are $\eta_\pm = d_\pm(\cdot, 0) \cup d_\pm(1, \cdot)$, and the normal vector field is just the restriction of v_\pm .

Consider a sutured Heegaard diagram $\mathcal{H}_{M_0} = (\Sigma_{M_0}, \alpha_{M_0}, \beta_{M_0})$ for the sutured manifold (M_0, γ_0) , and an extension of it to a sutured Heegaard diagram $\mathcal{H}_{M_1} = (\Sigma_{M_1}, \alpha_{M_1}, \beta_{M_1})$ for (M_1, γ_1) that are compatible with the contact structure ξ_W as in the definition of the gluing map; cf. [HKM08]. Choose analogous sutured Heegaard diagrams for (N_0, ν_0) and (N_1, ν_1) . The gluing maps are then defined at the complex level as the tensor product with contact classes, denoted by \mathbf{x}_M'' and \mathbf{x}_N'' .

We can suppose that the arcs η_\pm miss the α - and the β -curves by performing a finger move on the α - and β -curves along η_\pm . Thus, we can suppose that $\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}$ is subordinate to the bouquet $B(\mathbb{A}(0))$ and $\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}$ is subordinate to the bouquet $B'(\mathbb{A}(0))$. Furthermore, the sutured diagrams $\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}$ and $\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}$ are contact-compatible, and that the contact class is just $\mathbf{x}_M'' \otimes \mathbf{x}_N''$. Therefore, we have the commutativity of the diagram below:

$$\begin{array}{ccc} \mathrm{SFH}(\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}) & \xrightarrow{\varphi_{\mathcal{H}_{M_0}, \mathcal{H}_{N_0}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_0}) \otimes \mathrm{SFH}(\mathcal{H}_{N_0}) \otimes V \\ \downarrow \Phi_{\xi_W \sqcup \xi_U} & & \downarrow \Phi_{\xi_W} \otimes \Phi_{\xi_U} \otimes \mathrm{id}_V \\ \mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V. \end{array}$$

We are now left with the connected sum cobordism $\mathrm{id}_{M_1} \#_{\mathbb{A}} \mathrm{id}_{N_1}$, which still contains a bouquet, induced by $B(\mathbb{A})$. It follows from Case 1 that we have a commutative diagram as below, which concludes the proof in the case of boundary cobordisms:

$$\begin{array}{ccc} \mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V \\ \downarrow F_{\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}, \mathcal{H}_{M_1} \#_{\mathbb{A}(1)} \mathcal{H}_{N_1}} & & \downarrow \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_{M_1})} \otimes \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_{N_1})} \otimes \mathrm{id}_V \\ \mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(1)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(1)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V. \end{array}$$

Case 3: \mathcal{W} and \mathcal{U} are special cobordisms. Consider the function t on $B(\mathbb{A})$, and extend it to a Morse function on $\mathcal{W} \sqcup \mathcal{U}$. This gives a decomposition of the cobordisms \mathcal{W} and \mathcal{U} into 1-, 2-, and 3-handles that are attached *away from the bouquet*. Furthermore, $B(\mathbb{A})$ is a product with respect to this handle decomposition. In particular, $\mathbb{A}(0)$ and $B(\mathbb{A})_0$ conicide with $\mathbb{A}(1)$ and $B(\mathbb{A})_1$ after the handle attachments.

For each handle attachment, one can choose adapted Heegaard diagrams that are also subordinate to the bouquet $B(\mathbb{A})_0$ for the framed pair of points $\mathbb{A}(0)$. We therefore obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{SFH}(M_0 \#_{\mathbb{A}(0)} N_0) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \mathrm{SFH}(M_0) \otimes \mathrm{SFH}(N_0) \otimes V \\
\downarrow F_{\mathcal{W}}^s \#_{\mathbb{A}} \mathcal{U} & & \downarrow F_{\mathcal{W}}^s \otimes F_{\mathcal{U}}^s \otimes \mathrm{id}_V \\
\mathrm{SFH}(M_1 \#_{\mathbb{A}(0)} M_1) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \mathrm{SFH}(M_1) \otimes \mathrm{SFH}(N_1) \otimes V.
\end{array}$$

This concludes the proof of Theorem 5.9. \square

6. A $(1+1)$ -DIMENSIONAL TQFT

The aim of this section is to determine the $(1+1)$ -dimensional TQFT defined by the cobordism maps computed in Section 4.

6.1. The category of marked embedded cobordisms and the HFL TQFT.

We first define the cobordism category that we are interested in.

Definition 6.1. We define the category of *marked decorated embedded cobordisms* \mathbf{EmbCob}^* as follows. The objects are the standard n -component unlinks (U_n, P_{U_n}) for every $n \in \mathbb{N}$ with two decorations on each component. As Kronheimer and Mrowka suggest in [KM11, Section 8.2], we can take a specific model for it: We define U_n to be the union of standard circles L_1, \dots, L_n in the (x, y) -plane, each of diameter 0.5, centered at the first n integer lattice points along the x -axis, and such that $R_-(P_{U_n}) \cap L_i = L_i \cap \{x \leq i\}$. Furthermore, we suppose that the component L_1 of the unlink U_n is marked (in addition to being decorated), and denote it by a dot on the component itself.

The morphisms are all cobordisms in $S^3 \times I$ generated by

- V_n^e ($n \geq 1$), the cobordism from (U_n, P_{U_n}) to $(U_{n+1}, P_{U_{n+1}})$ obtained by stacking horizontally the cobordism V in Figure 15 and the identity cobordisms of $(U_{n-1}, P_{U_{n-1}})$. Note that the marked components are in the V part of the cobordism.
- Λ_n^e ($n \geq 2$), the cobordism from (U_n, P_{U_n}) to $(U_{n-1}, P_{U_{n-1}})$ obtained by stacking horizontally the cobordism Λ in Figure 15 and the identity cobordisms of $(U_{n-2}, P_{U_{n-2}})$. Note that the marked components are in the Λ part of the cobordism.
- $\mathrm{IX}_{i,n}^e$ (for $2 \leq i \leq n-1$ and $n \geq 3$), obtained by swapping the i -th and the $(i+1)$ -th component of (U_n, P_{U_n}) . Note that the marked components are on the first component of the cobordism, and that therefore the marked component cannot be swapped.
- IV_n^e ($n \geq 2$), the cobordism from (U_n, P_{U_n}) to $(U_{n+1}, P_{U_{n+1}})$ obtained by stacking horizontally an identity cobordism I between the marked components, the cobordism V and, finally, identity cobordisms on the last $n-2$ components.
- $\mathrm{I}\Lambda_n^e$ ($n \geq 3$), the cobordism from (U_n, P_{U_n}) to $(U_{n-1}, P_{U_{n-1}})$ obtained by stacking horizontally an identity cobordism I between the marked components, the cobordism Λ and, finally, identity cobordisms on the last $n-3$ components.

The above cobordisms are represented in Figure 19.

Our aim is to describe the TQFT

$$\mathbf{HFL} : \mathbf{EmbCob}^* \longrightarrow \mathbf{Vect}_{\mathbb{F}_2}.$$

To do so, we need to deal with some technicalities due to the fact that link Floer homology does not naturally arise as an invariant of a marked link. For this reason, we now explain how marking a component of (U_n, P_{U_n}) gives an isomorphism

$$(12) \quad \psi_{U_n} : \mathbf{HFL}(U_n, P_{U_n}) \xrightarrow{\cong} V^{\otimes(n-1)}.$$

As usual, let V be the \mathbb{F}_2 vector space of dimension 2, generated by two homogeneous elements T (top-graded) and B (bottom-graded). We now construct a canonical isomorphism

$$\mathbf{HFL}(U_n) \cong V^{\otimes(n-1)},$$

where each factor is associated to an unmarked component of U_n . Note that a sutured Heegaard diagram for the unknot in S^3 is given by $(A, \emptyset, \emptyset)$, where A is an annulus. We can construct a (sutured) Heegaard diagram for (U_n, P_{U_n}) by taking n copies of this Heegaard diagram (each one associated to a link component), and by connecting the first annulus (i.e., the one that corresponds to the marked component) to all the other annuli with tubes. Each tube contains a homotopically non-trivial α -curve and a homotopically non-trivial β -curve that intersect in two points, which we call B and T (for bottom-graded and top-graded). Then we have an isomorphism

$$\mathbf{HFL}(U_n, P_{U_n}) \xrightarrow{\cong} V_2 \otimes \dots \otimes V_n,$$

where by V_i we mean the vector space V associated to the tube connecting the first component to the i -th component.

Having set the isomorphism in equation (12), our next aim is to understand the cobordism maps induced by the generators of \mathbf{EmbCob}^* with respect to the standard basis consisting of elements of the form

$$v_2 \otimes \dots \otimes v_n,$$

where $v_i \in \{B, T\}$ for every $i \in \{2, \dots, n\}$.

6.1.1. The cobordism V_n^e . This cobordism is the disjoint union of the cobordism V and the identity cobordism $I^{\otimes(n-1)} = \text{id}_{U_{n-1}}$. Consider a framed pair of arcs \mathbb{A} such that the arc in V is close to the surface and its endpoints are close to the marked component of the link. Then by, Corollary 5.4, there are isomorphisms

$$\varphi_{U_n} : \mathbf{HFL}(U_n, P_{U_n}) \rightarrow \mathbb{F}_2 \otimes V^{\otimes(n-2)} \otimes V, \text{ and}$$

$$\varphi_{U_{n+1}} : \mathbf{HFL}(U_{n+1}, P_{U_{n+1}}) \rightarrow V \otimes V^{\otimes(n-2)} \otimes V$$

such that the cobordism map can be easily expressed with respect to these isomorphisms as in Theorem 5.9. However, φ_{U_n} and $\varphi_{U_{n+1}}$ are not the isomorphisms ψ_{U_n} and $\psi_{U_{n+1}}$ given in equation (12) for decorated unlinks with a marked component.

Choose Heegaard diagrams giving the splittings φ_{U_n} and $\varphi_{U_{n+1}}$ above. Since one of the two points in $\mathbb{A}(0)$ and $\mathbb{A}(1)$ is close to the marked component, the connecting tube (to which the last V factor is associated) can be supposed to connect the first (i.e., the marked) component of the unlink to a Heegaard diagram for $(U_{n-1}, P_{U_{n-1}})$ or (U_n, P_{U_n}) . It follows that we can change the given Heegaard diagrams for U_n

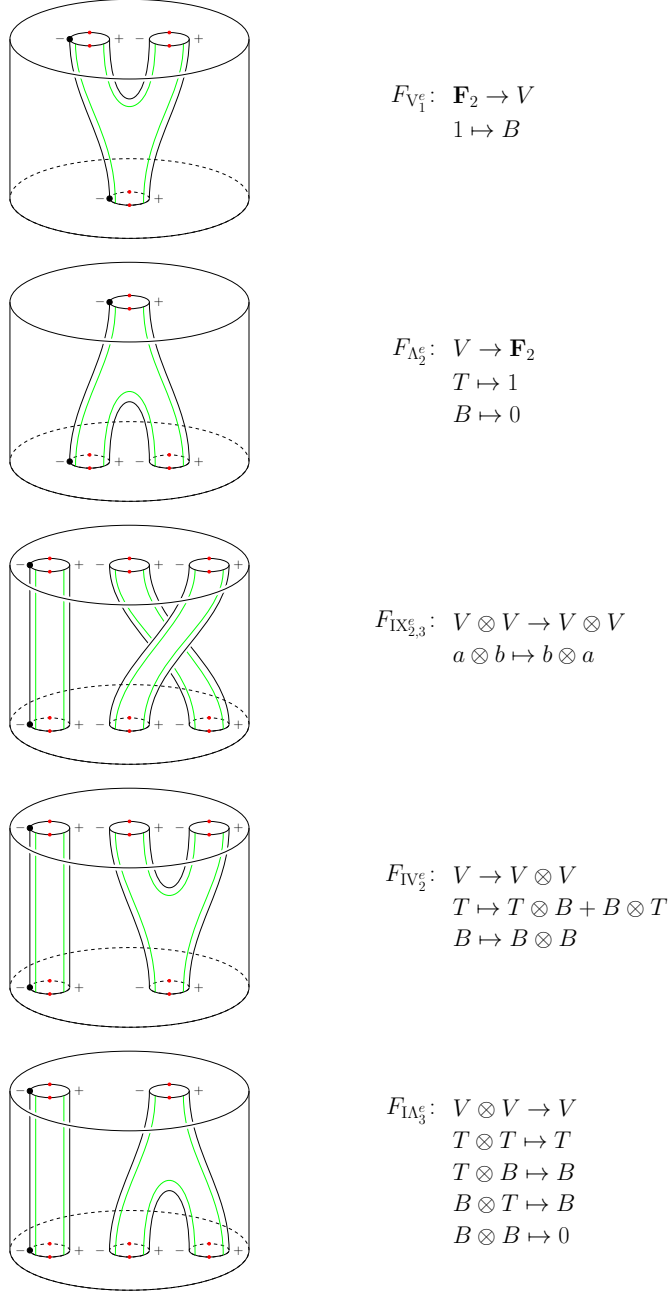


FIGURE 19. The figure shows the cobordisms V_1^e , Λ_2^e , $IX_{2,3}^e$, IV_2^e and $I\Lambda_3^e$ in the category \mathbf{EmbCob}^* (represented following the conventions explained in Figure 8), and the maps they induce via the TQFT HFL. Note that the marked component allows us to choose a canonical identification $\psi_{U_n}: \text{HFL}(U_n, P_{U_n}) \rightarrow V^{\otimes(n-1)}$.

and U_{n+1} , which we denote by \mathcal{H} , to the standard ones \mathcal{H}_{std} that give the isomorphisms ψ_{U_n} and $\psi_{U_{n+1}}$ in such a way that the naturality map affects only the last factors $V^{\otimes(n-2)}$ and V . Therefore, if we consider the commutative diagram

$$\begin{array}{ccc}
\text{HFL}(U_n, P_{U_n}) & \xrightarrow{\psi_{U_n}} & \mathbb{F}_2 \otimes V^{\otimes(n-1)} \\
\downarrow \text{id} & & \downarrow \text{id}_{\mathbb{F}_2} \otimes F_{\mathcal{H}_{\text{std}}, \mathcal{H}} \\
\text{HFL}(U_n, P_{U_n}) & \xrightarrow{\varphi_{U_n}} & \mathbb{F}_2 \otimes (V^{\otimes(n-2)} \otimes V) \\
\downarrow F_{V_n^e} & & \downarrow F_V \otimes \text{id}_{V^{\otimes(n-2)}} \otimes \text{id}_V \\
\text{HFL}(U_{n+1}, P_{U_{n+1}}) & \xrightarrow{\varphi_{U_{n+1}}} & V \otimes (V^{\otimes(n-2)} \otimes V) \\
\downarrow \text{id} & & \downarrow \text{id}_V \otimes F_{\mathcal{H}_{\text{std}}, \mathcal{H}}^{-1} \\
\text{HFL}(U_{n+1}, P_{U_{n+1}}) & \xrightarrow{\psi_{U_{n+1}}} & V \otimes V^{\otimes(n-1)},
\end{array}$$

we see that the map $F_{V_n^e}$ with respect to the isomorphisms ψ_{U_n} and $\psi_{U_{n+1}}$ is

$$\begin{array}{ccc}
F_{V_n^e} : \mathbb{F}_2 \otimes V^{\otimes(n-1)} & \longrightarrow & V \otimes V^{\otimes(n-1)}. \\
1 \otimes * & \longmapsto & B \otimes *
\end{array}$$

6.1.2. *The cobordism Λ_n^e .* This cobordism is the disjoint union of the merge cobordism Λ as in Figure 15, from the first two components of U_n to the first component of U_{n-1} , and the identity cobordism on the last $n-2$ components. By arguing as in Section 6.1.1, we see that the map $F_{\Lambda_n^e}$ with respect to the isomorphisms ψ_{U_n} and $\psi_{U_{n-1}}$ is

$$\begin{array}{ccc}
F_{\Lambda_n^e} : V \otimes V^{\otimes(n-2)} & \longrightarrow & \mathbb{F}_2 \otimes V^{\otimes(n-2)}. \\
T \otimes * & \longmapsto & 1 \otimes * \\
B \otimes * & \longmapsto & 0
\end{array}$$

6.1.3. *The cobordism $\text{IX}_{i,n}^e$.* The effect of the cobordism $\text{IX}_{i,n}^e$ on the Heegaard diagram that induces the isomorphism ψ_{U_n} in equation (12) is a swap between the two (unmarked) components i and $i+1$. The map on HFL is therefore

$$\begin{array}{ccc}
F_{\text{IX}_{i,n}^e} : V^{\otimes(i-2)} \otimes V \otimes V \otimes V^{\otimes(n-i-1)} & \longrightarrow & V^{\otimes(i-2)} \otimes V \otimes V \otimes V^{\otimes(n-i-1)}. \\
* \otimes a \otimes b \otimes * & \longmapsto & * \otimes b \otimes a \otimes *
\end{array}$$

6.1.4. *The cobordism IV_n^e .* We now turn our attention to the cobordism IV_n^e . By Theorem 5.9, and arguing as in Section 6.1.1, we can restrict to the case $n=2$. Then we can split the cobordism IV_2^e as the disjoint union of the identity cobordism I and

the split cobordism V from Figure 15. By Theorem 5.9, we have isomorphisms

$$\begin{aligned}\varphi_{U_2}: \text{HFL}(U_2, P_{U_2}) &\rightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V, \text{ and} \\ \varphi_{U_3}: \text{HFL}(U_3, P_{U_3}) &\rightarrow \mathbb{F}_2 \otimes V \otimes V,\end{aligned}$$

such that the map $F_{\text{IV}_2^e}$ under these identifications is as follows:

$$\begin{aligned}F_{\text{IV}_2^e}: \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V &\longrightarrow \mathbb{F}_2 \otimes V \otimes V. \\ 1 \otimes 1 \otimes T &\longmapsto 1 \otimes B \otimes T \\ 1 \otimes 1 \otimes B &\longmapsto 1 \otimes B \otimes B\end{aligned}$$

Recall that we are however interested in writing the map with respect to identifications in equation (12). Notice that the map

$$\varphi_{U_2} \circ \psi_{U_2}^{-1}: V \rightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V$$

is clear because it preserves the relative grading. We can therefore identify

$$\text{HFL}(U_2, P_{U_2}) \cong V \cong \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V.$$

We need to understand $\psi_{U_3} \circ F_{\text{IV}_2^e}(T)$ and $\psi_{U_3} \circ F_{\text{IV}_2^e}(B)$. Notice that $\varphi_{U_3} \circ F_{\text{IV}_2^e}(B)$ is the unique bottom-graded element of $\mathbb{F}_2 \otimes V \otimes V$, therefore we deduce that

$$\psi_{U_3} \circ F_{\text{IV}_2^e}(B) = B \otimes B \in V \otimes V.$$

On the other hand, T is mapped to one of the three middle-graded homogeneous elements of $\mathbb{F}_2 \otimes V \otimes V$, so $\psi_{U_3} \circ F_{\text{IV}_2^e}(T)$ is also one of the middle-graded homogeneous elements: $T \otimes B$, $B \otimes T$, or $T \otimes B + B \otimes T$. If we consider the composition of cobordisms $\text{IX}_{2,3}^e \circ \text{IV}_2^e$, we obtain the cobordism IV_2^e with *different decorations*, which by Remark 4.1 induces the same map as IV_2^e on HFL . Therefore, we have that the map $\psi_{U_3} \circ F_{\text{IV}_2^e}$ is invariant under swap. In the end, the map associated to $F_{\text{IV}_n^e}$ with respect to the identifications in equation (12) is

$$\begin{aligned}F_{\text{IV}_n^e}: V \otimes V^{\otimes(n-2)} &\longrightarrow V^{\otimes 2} \otimes V^{\otimes(n-2)}. \\ T \otimes * &\longmapsto (T \otimes B + B \otimes T) \otimes * \\ B \otimes * &\longmapsto (B \otimes B) \otimes *\end{aligned}$$

6.1.5. *The cobordism $\text{I}\Lambda_n^e$.* The last case that we have to study is the cobordism $\text{I}\Lambda_n^e$. As in Section 6.1.4, we can restrict our attention to $\text{I}\Lambda_3^e$.

By Theorem 5.9, we have isomorphisms φ_{U_2} and φ_{U_3} as in Section 6.1.4 such that the map $F_{\text{I}\Lambda_3^e}$ under these identifications is as follows:

$$\begin{aligned}F_{\text{I}\Lambda_3^e}: \mathbb{F}_2 \otimes V \otimes V &\longrightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V. \\ 1 \otimes T \otimes T &\longmapsto 1 \otimes 1 \otimes T \\ 1 \otimes T \otimes B &\longmapsto 1 \otimes 1 \otimes B \\ 1 \otimes B \otimes T &\longmapsto 0 \\ 1 \otimes B \otimes B &\longmapsto 0\end{aligned}$$

As in Section 6.1.4, we need to understand this map with respect to the isomorphisms given by equation (12). By the fact that $\text{IA}_3^\epsilon \circ \text{IX}_{2,3}^\epsilon$ is the cobordism IA_3^ϵ (with different decorations) and by Remark 4.2, we know that the map $F_{\text{IA}_3^\epsilon}$ with respect to such identifications is invariant under swaps. Since we know that there is a unique middle-graded homogeneous element of $V^{\otimes 2}$ that is in the kernel of $F_{\text{IA}_3^\epsilon}$, and since $\ker(F_{\text{IA}_3^\epsilon})$ is invariant under swaps, we deduce that such an element is $T \otimes B + B \otimes T$. Therefore, the map $F_{\text{IA}_n^\epsilon}$ with respect to the identifications in equation (12) is given by

$$\begin{aligned} F_{\text{IA}_n^\epsilon} : V^{\otimes 2} \otimes V^{\otimes(n-3)} &\longrightarrow V \otimes V^{\otimes(n-3)}. \\ (T \otimes T) \otimes * &\longmapsto T \otimes * \\ (T \otimes B) \otimes * &\longmapsto B \otimes * \\ (B \otimes T) \otimes * &\longmapsto B \otimes * \\ (B \otimes B) \otimes * &\longmapsto 0 \end{aligned}$$

Remark. The results above completely determine the TQFT HFL. This is expressed in a concise form in Figure 19.

6.2. The category of marked abstract cobordisms and the reduced Khovanov TQFT. We now compare the TQFT HFL from Section 6.1 with another TQFT from a category of marked cobordisms, namely the reduced Khovanov TQFT, which gives rise to the reduced Khovanov homology of a marked link. We start with the following definition.

Definition 6.2. We define the category of *marked abstract cobordisms* \mathbf{AbsCob}^* as follows. The objects are closed 1-manifolds with a marked component, which we will denote by a dot on the component itself. Let U_n be a particular closed 1-manifold with n components; we will suppose that each component is labeled by a number between 1 and n , and that the marked component is the first one. Notice that any object of \mathbf{AbsCob}^* is diffeomorphic to some U_n .

The morphisms are all abstract cobordisms between any two closed 1-manifolds generated by

- V_n^a ($n \geq 1$), the cobordism from U_n to U_{n+1} which consists of a pair-of-pants between the first component of U_n and the first two components of U_{n+1} , and, for each $i > 1$, of a cylinder between the i -th component of U_n and the $(i+1)$ -th component of U_{n+1} . Notice that the marked components of U_n and U_{n+1} are both contained in the pair-of-pants.
- Λ_n^a ($n \geq 2$), the cobordism from U_n to U_{n-1} which consists of a pair-of-pants between the first two components of U_n and the first component of U_{n-1} , and, for each $i > 2$, of a cylinder between the i -th component of U_n and the $(i-1)$ -th component of U_{n-1} . Notice that the marked components of U_n and U_{n-1} are both contained in the pair-of-pants.
- $\text{IX}_{i,n}^a$ (for $2 \leq i \leq n-1$ and $n \geq 3$), obtained by swapping the i -th and the $(i+1)$ -th component of U_n . Notice that the marked components are on the

first I component of the cobordism, and that therefore the marked component cannot be swapped.

- IV_n^a ($n \geq 2$), the cobordism from U_n to U_{n+1} which consists of a pair-of-pants between the second component of U_n and the second and the third components of U_{n+1} , and of cylinders between all the other components. The marked components of U_n and U_{n+1} are connected by a cylinder.
- IA_n^a ($n \geq 3$), the cobordism from U_n to U_{n-1} which consists of a pair-of-pants between the second and the third components of U_n and the second component of U_{n-1} , and of cylinders between all the other components. The marked components of U_n and U_{n-1} are connected by a cylinder.

The above cobordisms are represented in Figure 20.

Definition 6.3. Let

$$\text{Ob}: \mathbf{EmbCob}^* \rightarrow \mathbf{AbsCob}^*,$$

be the forgetful functor such that on the objects $\text{Ob}(U_n, P_{U_n}) = U_n$, and on the cobordisms it simply forgets all the decorations and the embedding.

Definition 6.4. The reduced Khovanov TQFT is the functor

$$\widetilde{\text{Kh}}: \mathbf{AbsCob}^* \rightarrow \mathbf{Vect}_{\mathbb{F}_2}$$

such that on the objects $\widetilde{\text{Kh}}(U_n) = V^{\otimes(n-1)}$, where each V factor should be thought of as associated to an unmarked component, and on the morphisms it is defined as in Figure 20.

The main result of the section is expressed in the following theorem.

Theorem 6.5. *There is a commutative triangle of functors*

$$\begin{array}{ccc} \mathbf{EmbCob}^* & \xrightarrow{\text{HFL}} & \mathbf{Vect}_{\mathbb{F}_2} \\ & \searrow \text{Ob} & \nearrow \widetilde{\text{Kh}} \\ & \mathbf{AbsCob}^* & \end{array}$$

Proof. We have by definition

$$\text{HFL}(U_n, P_{U_n}) = V^{\otimes(n-1)} = \widetilde{\text{Kh}}(U_n) = \widetilde{\text{Kh}}(\text{Ob}(U_n, P_{U_n})).$$

The commutativity of the triangle on the morphisms can be checked just on the generators of the morphisms. This is achieved by comparing Figures 19 and 20. \square

As a consequence of Theorem 6.5, we notice that the reduced Khovanov homology of L can be computed by using the HFL TQFT. See Section 7 for the definition of *cube of resolutions*.

Corollary 6.6. *The reduced Khovanov homology (with \mathbb{F}_2 coefficients) of a marked link L can be computed by applying the TQFT HFL to a cube of resolutions of L .*

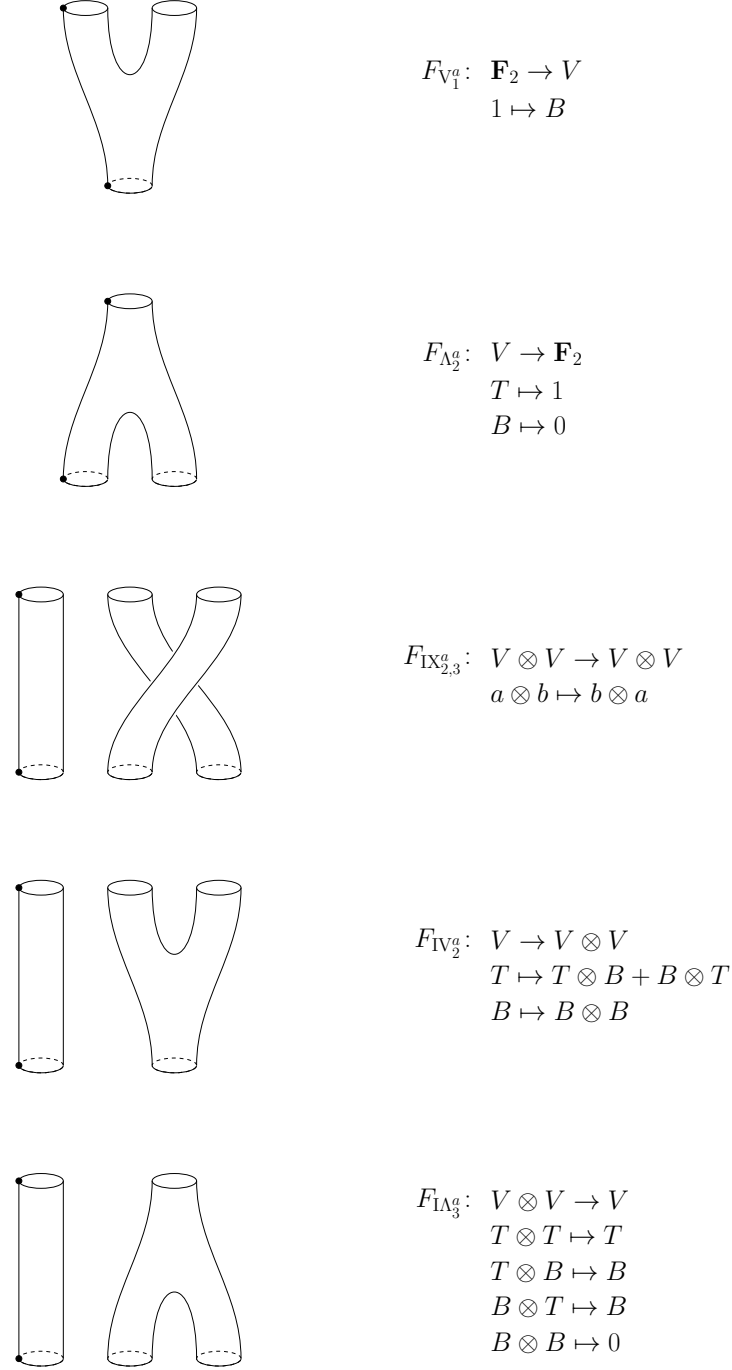


FIGURE 20. The figure shows the cobordisms V_1^a , Λ_2^a , $IX_{2,3}^a$, IV_2^a and IA_3^a in the category \mathbf{AbsCob}^* and the maps they induce via the reduced Khovanov TQFT $\widetilde{\text{Kh}}$. Note that the marked component gives a canonical identification $\widetilde{\text{Kh}}(U_n) = V^{\otimes(n-1)}$.

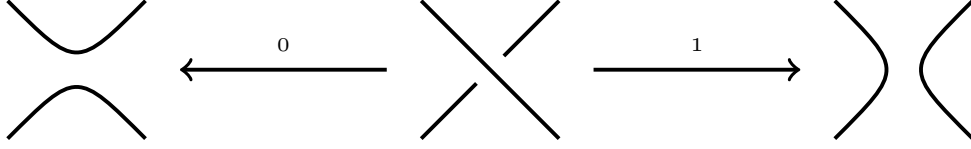


FIGURE 21. The figure shows the 0-smoothing and the 1-smoothing for a crossing of an unoriented link.

7. A SPECTRAL SEQUENCE FROM KHOVANOV HOMOLOGY

In this last section, we note that one can define a spectral sequence from Khovanov homology (or reduced Khovanov homology) by using the cobordism maps as higher differentials. We prove that the spectral sequence is invariant under Reidemeister moves and that it is therefore a link invariant up to isomorphism.

Such a spectral sequence has also been independently studied by Baldwin, Hedden, and Lobb. Their recent paper [BHL15] develops a technique to study “Khovanov-Floer” theories that can be used to prove the invariance of the spectral sequence that we are going to define. Moreover, their proof gives results also in terms of functoriality.

We first recall the definition of a cube of resolutions for a link diagram.

Definition 7.1. Let L be a marked link in S^3 , and let D be a marked diagram for L . A *resolution* of D is a map u from the set of crossings of D to $\{0, 1\}$.

If u is a resolution, we denote by L_u the marked unlink obtained by smoothing each crossing according to the conventions in Figure 21, which we also call *resolution*.

If u and v are two resolutions of D , we say that $u < v$ if v is obtained by changing the value of some crossings of u from 0 to 1.

Definition 7.2. If $u < v$ and u and v only differ at a single crossing, then L_v is obtained from L_u by a pair-of-pants cobordism. We denote it by $\mathcal{G}_{u,v}$.

If $u < v$ and u and v differ at k crossings, then choose a sequence $u = w_0 < \dots < w_k = v$, where w_i and w_{i+1} differ only at one crossing. Let $\mathcal{G}_{u,v}$ denote the composition of the cobordisms $\mathcal{G}_{w_{k-1}, w_k} \circ \dots \circ \mathcal{G}_{w_0, w_1}$. Such a cobordism is independent of the choice of the intermediate resolutions, up to equivalence.

We say that the family of all resolutions L_u together with the cobordisms $\mathcal{G}_{u,v}$ such that $u < v$ and u and v only differ at a single crossing is a *cube of resolutions* for the marked diagram D .

We say that the family of all resolutions L_u , together with the cobordisms $\mathcal{G}_{u,v}$ is a *full cube of resolutions* for the marked diagram D .

We now want to apply the TQFT HFL defined in Section 6 to a full cube of resolutions for a diagram of a link L in S^3 .

Definition 7.3. Let L be a marked link in S^3 , together with a diagram D . We define a complex associated to its full cube of resolutions as follows. Given a resolution u , let P_{L_u} be a decoration of L_u that consists of two points on each component of L_u . Since (L_u, P_{L_u}) and $(U_{|L_u|}, P_{U_{|L_u|}})$ are diffeomorphic as marked decorated links, we

have an isomorphism

$$\mathrm{HFL}(L_u, P_{L_u}) \cong \mathrm{HFL}(U_{|L_u|}, P_{U_{|L_u|}}).$$

To every resolution L_u , we associate the vector space

$$\tilde{C}_u = \mathrm{HFL}(L_u, P_{L_u}).$$

Let \tilde{C} be the \mathbb{F}_2 vector space

$$\tilde{C} = \bigoplus_{\text{all resolutions } u} \tilde{C}_u.$$

If $u < v$, let $\tilde{\partial}_{u,v} : \tilde{C}_u \rightarrow \tilde{C}_v$ denote the map induced by the cobordism $\mathcal{G}_{u,v}$ on HFL . Let the map $\tilde{\partial}$ be

$$\tilde{\partial} = \sum_{u < v} \tilde{\partial}_{u,v}.$$

Lemma 7.4. *The map $\tilde{\partial}$ is a differential; i.e., $\tilde{\partial}^2 = 0$.*

Proof. The proof is immediate as we are working with \mathbb{F}_2 coefficients. Notice that

$$\tilde{\partial}^2 = \sum_{w < z} \tilde{\partial}_{w,z} \circ \sum_{u < v} \tilde{\partial}_{u,v} = \sum_{u < v = w < z} \tilde{\partial}_{w,z} \circ \tilde{\partial}_{u,v} = \sum_{u < v = w < z} \tilde{\partial}_{u,z} = \sum_{u < z} \sum_{\substack{v \\ u < v < z}} \tilde{\partial}_{u,z}.$$

The set $\{v : u < v < z\}$ has cardinality $2^k - 2$, where k is the number of crossings where u is different from z . Since we are working over \mathbb{F}_2 , we immediately obtain that $\tilde{\partial}^2 = 0$. \square

The complex $(\tilde{C}, \tilde{\partial})$ is also filtered, as we explain in the next definition.

Definition 7.5. Let L be a marked link in S^3 , together with a marked diagram D . Consider the complex $(\tilde{C}, \tilde{\partial})$ from Definition 7.3. For $i \in \mathbb{Z}$, let

$$\tilde{C}_i = \bigoplus_{\substack{u \text{ resolution with} \\ i \text{ 1-smoothings}}} \tilde{C}_u,$$

and define the filtration

$$(13) \quad \mathcal{F}_p(\tilde{C}) = \bigoplus_{i \geq p} \tilde{C}_i.$$

Then $(\bigoplus_{i \in \mathbb{Z}} \tilde{C}_i, \tilde{\partial})$ is a graded filtered complex, which we call the *reduced Khovanov filtered complex* associated to the marked diagram D .

Strictly speaking, the complex $(\tilde{C}, \tilde{\partial})$ is not graded, because the differential does not respect the grading. However, we use the term *graded filtered* to underline the fact that the filtration comes from a grading on \tilde{C} ; cf. Equation (13).

As explained for instance in [McC01], a filtered complex yields a spectral sequence whose E^2 page is the homology of the associated graded complex. Hence, by Corollary 6.6, in our case the E^2 page is the reduced Khovanov homology of L .

Remark. The definition of the reduced Khovanov filtered complex actually does not require any Floer homology. It can also be defined using the formalism of Bar-Natan [BN02] or Khovanov [Kho00]. The vector space \tilde{C}_u can be defined as

$$\tilde{C}_u = V^{\otimes |L_u|} / \left(\langle v_- \rangle \otimes V^{\otimes (|L_u|-1)} \right),$$

where the v_- is associated to the marked component of L_u . The cobordism maps are given by the reduced Khovanov TQFT, which we can denote by either $\widetilde{\text{Kh}}$ or HFL, and whose definition does not require any Floer theory.

In a similar way as above, we can also define an unreduced *Khovanov filtered complex* for a link diagram by applying the usual Khovanov TQFT [BN02] to the full cube of resolutions in Definition 7.2.

Definition 7.6. Let L be a link in S^3 , together with a diagram D . We define a complex associated to its full cube of resolutions as follows. To every resolution L_u associate the vector space $C_u \cong V^{\otimes |L_u|}$, where $|L_u|$ denotes the number of components of L_u . Let C be the \mathbb{F}_2 vector space

$$C = \bigoplus_{\text{all resolutions } r} C_r.$$

If $u < v$, let the map $\partial_{u,v} : C_u \rightarrow C_v$ denote the map induced by applying the Khovanov TQFT to the cobordism $\mathcal{G}_{u,v}$. The map

$$\partial = \sum_{u < v} \partial_{u,v}$$

is a differential on C , that is $\partial^2 = 0$.

Again, the complex (C, ∂) comes with a filtration.

Definition 7.7. Let L be a link in S^3 , together with a diagram D . Consider the complex (C, ∂) from Definition 7.6. For $i \in \mathbb{Z}$, let

$$C_i = \bigoplus_{\substack{u \text{ resolution with} \\ i \text{ 1-smoothings}}} C_u,$$

and define the filtration

$$\mathcal{F}_p(C) = \bigoplus_{i \geq p} C_i.$$

Then $(\bigoplus_{i \in \mathbb{Z}} C_i, \partial)$ is a graded filtered complex, which we call the *Khovanov filtered complex* associated to the diagram D .

The E^2 page of the spectral sequence arising from the Khovanov filtered complex is by definition the Khovanov homology of L .

It follows from [BHL15] that the spectral sequences above are (marked) link invariants. We give here an elementary proof of the fact that the spectral sequences are (marked) link invariants *up to isomorphism*, based on the proof of the invariance of Khovanov homology under Reidemeister moves [BN02].

Theorem 1.2. *The spectral sequence defined by the Khovanov filtered complex is an invariant of the link up to isomorphism.*

Proof. What we need to check is that for each Reidemeister move (R1), (R2), and (R3), as defined in [BN02], there is a morphism of filtered chain complexes that induces an isomorphism on all pages.

Let (C, ∂) denote the Khovanov filtered complex. The grading allows us to split the differential as

$$\partial = \partial_1^h + \partial_2^h + \dots,$$

where ∂_i^h denotes the component of the differential ∂ which is homogeneous of degree i . We also define

$$\partial|_i = \partial_1^h + \partial_2^h + \dots + \partial_i^h.$$

Notice that, with the above notation, $H^*(C, \partial|_i) \cong E^{i+1}$, and the differential ∂_{i+1} on E^{i+1} is induced by $\partial|_{i+1}$. Recall that, as noticed before, $E^2 \cong \text{Kh}(L)$, the Khovanov homology of the link.

We start the proof by stating the following analogue of [BN02, Lemma 3.7]. We say that a subcomplex $C' \subseteq C$ of a graded filtered complex is a *graded filtered subcomplex* if $C' = \bigoplus_{i \in \mathbb{Z}} C'_i$, where $C'_i \subseteq C_i$, and $\partial(C') \subseteq C'$.

Lemma 7.8. *Let $C' \subseteq C$ be a graded filtered subcomplex of a graded filtered complex (C, ∂) , and suppose that $H^*(C', \partial|_1) = 0$ (we say that C' is $\partial|_1$ -acyclic). Then C/C' is also a graded filtered complex, and the quotient map $C \rightarrow C/C'$ induces an isomorphism of spectral sequences.*

Proof. We define a grading on C/C' by setting

$$(C/C')_i = C_i/C'_i.$$

The differential ∂ on C induces a differential $\bar{\partial}$ on C/C' , which respects the filtration and can be decomposed into homogeneous components

$$\bar{\partial} = \bar{\partial}_1^h + \bar{\partial}_2^h + \dots$$

As before, we define the maps

$$\bar{\partial}|_i = \bar{\partial}_1^h + \bar{\partial}_2^h + \dots + \bar{\partial}_i^h.$$

The fact that the filtrations on C and C' come from a grading implies that $\bar{\partial}|_i$ coincides with the map induced by $\partial|_i: C \rightarrow C$ on the quotient, which we denote by $\bar{\partial}|_i$.

The fact that C' is $\partial|_1$ -acyclic implies that it is $\partial|_i$ -acyclic for all $i \geq 1$. Then the long exact sequence associated to the inclusion of complexes $(C', \partial|_i) \subseteq (C, \partial|_i)$ implies that the quotient map $C' \rightarrow C/C'$ induces isomorphisms $E_i(C) \rightarrow E_i(C/C')$ on all pages. \square

We now prove invariance under Reidemeister moves.

Invariance under (R1). Consider Bar-Natan's proof in [BN02, Section 3.5.1]. The subcomplex \mathcal{C}' defined by Bar-Natan is actually a graded filtered subcomplex of \mathcal{C} . The fact that \mathcal{C}' is $\partial|_1$ -acyclic was already observed there by Bar-Natan.

By Lemma 7.8, we can consider the graded filtered complex \mathcal{C}/\mathcal{C}' , which is isomorphic (up to a grading shift) to the graded filtered complex $\llbracket D \rrbracket$, where D is the diagram of the link without the curl arising from the Reidemeister move (R1). As in the proof by Bar-Natan, the grading shift is adjusted by defining the complex $\mathcal{C}(L)$ from $\llbracket L \rrbracket$.

Invariance under (R2). We follow the second proof of the invariance under (R2), from [BN02, Sections 3.5.2 and 3.5.4]. The complex \mathcal{C}' is again a graded filtered subcomplex. Bar-Natan proves that it is $\partial|_1$ -acyclic, so, by Lemma 7.8, we can focus on \mathcal{C}/\mathcal{C}' .

What we need to check next is that the subcomplex $\mathcal{C}''' \subseteq \mathcal{C}/\mathcal{C}'$, as defined by Bar-Natan, is a $\partial|_1$ -acyclic graded filtered complex. The map $d_{\star 0} \circ \Delta^{-1}$ is homogeneous, so \mathcal{C}''' is a graded subspace of \mathcal{C}/\mathcal{C}' . Bar-Natan already proved that \mathcal{C}''' is $\partial|_1$ -acyclic, so we only need to prove that it is closed under the differential ∂ . We need to check this on the elements of the form α or $(\beta, \tau(\beta))$ ¹.

- Consider an element of the form α in \mathcal{C}''' . Any component \mathcal{G} of the differential ∂ that does not contain the maps Δ and $d_{\star 0}$ sends α to some $\alpha' \in \mathcal{C}'''$. On the other hand, the other components of ∂ are sums of maps that can be written as $(\Delta + d_{\star 0}) \circ \mathcal{G}$, where \mathcal{G} is the composition of some other edge maps. As already seen, $\mathcal{G}(\alpha) = \alpha' \in \mathcal{C}'''$, and the fact that $(\Delta + d_{\star 0})(\alpha') \in \mathcal{C}'''$ was already observed by Bar-Natan.
- We now show that $\partial(\beta, \tau(\beta)) \in \mathcal{C}'''$. We prove that the subspace $\{(\beta, \tau(\beta))\}$ is invariant under any edge map. If \mathcal{G} is an edge map, we need to check that $\mathcal{G} \circ \tau = \tau \circ \mathcal{G}$. Since Δ is an isomorphism, this is equivalent to proving that $\mathcal{G} \circ \tau \circ \Delta = \tau \circ \mathcal{G} \circ \Delta$. The fact that Kh is a TQFT implies that the edge maps commute, so we have

$$\begin{aligned}
 \mathcal{G} \circ \tau \circ \Delta &= \mathcal{G} \circ (d_{\star 0} \circ \Delta^{-1}) \circ \Delta \\
 &= \mathcal{G} \circ d_{\star 0} \\
 &= d_{\star 0} \circ (\Delta^{-1} \circ \Delta) \circ \mathcal{G} \\
 &= (d_{\star 0} \circ \Delta^{-1}) \circ \mathcal{G} \circ \Delta \\
 &= \tau \circ \mathcal{G} \circ \Delta.
 \end{aligned}$$

Invariance under (R3). In the same way as we did for (R2), one can prove that the complexes \mathcal{C}' and \mathcal{C}''' are also $\partial|_1$ -acyclic graded filtered subcomplexes in this case. To conclude the proof, it is sufficient to note that, since the map Υ defined by Bar-Natan commutes with all edge maps, it also commutes with ∂ , so it is an isomorphism of graded filtered chain complexes. \square

¹ We use the notation of [BN02, Section 3.5.4].

7.1. Final remarks. The spectral sequence arising from the Khovanov filtered complex is an instance of a *Khovanov-Floer theory*, as defined by Baldwin, Hedden, and Lobb [BHL15]. Their theory not only implies that the spectral sequence is a link invariant up to isomorphism, but also achieves naturality and functoriality under link cobordisms.

A result analogous to Theorem 1.2 also holds for marked links and the reduced Khovanov filtered complex. The proof is obtained by adapting the proof of Theorem 1.2. First, notice that in order to go from a pointed diagram D to another pointed diagram D' of a marked link L , it is sufficient to use Reidemeister moves that do not cross the basepoint: As explained by Hedden and Ni [HN13, page 3032], when you encounter a Reidemeister move that crosses the basepoint, you can trade it for a sequence of Reidemeister moves that do not cross it by pulling the string in the other direction, and letting it pass over the point at infinity. Recall that the reduced Khovanov complex is defined by quotienting the Khovanov complex by $\langle v_- \rangle$, where the v_- is associated to the marked circle in each resolution. In order to check that the Reidemeister moves that do not cross the basepoint induce filtered chain isotopies between the reduced Khovanov complexes, we need to check the following two things:

- the maps of filtered complexes defined in the sections on the invariance under Reidemeister moves send $\langle v_- \rangle$ to $\langle v_- \rangle$;
- if $\widetilde{\mathcal{C}}'$ and $\widetilde{\mathcal{C}}'''$ denote the quotients of the complexes \mathcal{C}' and \mathcal{C}''' by $\langle v_- \rangle$, we need to check that $\widetilde{\mathcal{C}}'$ and $\widetilde{\mathcal{C}}'''$ are $\widetilde{\partial}|_1$ -acyclic.

Both points can be checked to be true for any Reidemeister move that happens away from the basepoint.

The spectral sequences above split along the quantum grading, because all the differentials that we consider are q -homogeneous. So, actually, one gets a spectral sequence for each q -grading, and this is a (marked) link invariant.

It is straightforward from the definition of the (reduced) Khovanov filtered complex that the spectral sequences above are trivial on Kh-thin knots. The page E^2 is indeed supported along two diagonals in the (p, q) grading, so there is no room for non-trivial higher differentials.

With \mathbb{F}_2 coefficients, the spectral sequence associated to the reduced Khovanov complex does not depend on the basepoint (the same happens for the usual reduced Khovanov homology). If u is a resolution of the diagram D , let $V(u)$ be the free vector space generated by the circles in the resolution. Call the obvious generators X_1, \dots, X_n . (Here, it is more convenient to use Khovanov's notation [Kho00]. The vector X_i can be translated to Bar-Natan's notation [BN02] as the v_- corresponding to the i -th circle of u .) At the u -vertex of the Khovanov complex one finds the vector space $\bigwedge V(u)$. (In Bar-Natan's notation $X_1 \wedge X_2 \wedge \dots \wedge X_j$ corresponds to $v_-^{(1)} \otimes \dots \otimes v_-^{(j)} \otimes v_+^{(j+1)} \otimes \dots \otimes v_+^{(n)}$.) If W is the subspace of V spanned by the differences (or sums) of the generators, then the subcomplex $\bigwedge W \subseteq \bigwedge V(u)$ is isomorphic to the usual reduced Khovanov complex $\bigwedge V(u)/\langle X_n \rangle$ (here, we assume that the marked component is the n -th one.) The isomorphism $\bigwedge V(u)/\langle X_n \rangle \rightarrow \bigwedge W$ is given by

$$X_{i_1} \wedge \dots \wedge X_{i_k} \mapsto (X_{i_1} - X_n) \wedge \dots \wedge (X_{i_k} - X_n).$$

This isomorphism preserves the filtration and commutes with the edge maps, so it induces an isomorphism of spectral sequences. This point of view has been communicated to us by Andrew Lobb, and was firstly observed by Ozsváth and Szabó [OSz05].

The fact that the spectral sequence is an invariant implies that, in particular, each page of the spectral sequence is a bigraded vector space $\mathrm{Kh}_{p,q}^r$ that is a link invariant for every $r \in \{2, 3, \dots, \infty\}$. The abutment of this spectral sequence is unknown. A trivial lower bound to the dimension of $\mathrm{Kh}_{*,*}^\infty$ is the number of non-zero coefficient of the Jones polynomial, but this is a rough bound. We are aware that Baldwin and Lobb ran a simulation to explore the limit of this spectral sequence, but could not find any higher differentials. We therefore conclude with their conjecture, based on their data.

Conjecture 1.3 (Baldwin-Lobb). *The limit $\mathrm{Kh}_{p,q}^\infty$ of the spectral sequence is Khovanov homology $\mathrm{Kh}_{p,q} = \mathrm{Kh}_{p,q}^2$.*

REFERENCES

- [AK10] G. Arone and M. Kankaanrinta, *On the functoriality of the blow-up construction*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), no. 5, 821–832.
- [Ati88] M. Atiyah, *Topological quantum field theories*, Publications Mathématiques de l’IHÉS **68** (1988), no. 1, 175–186.
- [BHL15] J. Baldwin, M. Hedden, and A. Lobb, *On the functoriality of Khovanov-Floer theories*, arXiv:1509.04691 (2015).
- [BN02] D. Bar-Natan, *On Khovanov’s categorification of the Jones polynomial*, Algebraic & Geometric Topology **2** (2002), no. 1, 337–370.
- [BT06] C. Blanchet and V. Turaev, *Axiomatic approach to Topological Quantum Field Theory*, Encyclopedia of Mathematical Physics (2006), 232–234.
- [Gab83] D. Gabai, *Foliations and the topology of 3-manifolds*, Journal of Differential Geometry **18** (1983), no. 3, 445–503.
- [HKM08] K. Honda, W. H. Kazez, and G. Matic, *Contact structures, sutured Floer homology and TQFT*, arXiv:0807.2431 (2008).
- [HN13] M. Hedden and Y. Ni, *Khovanov module and the detection of unlinks*, Geometry & Topology **17** (2013), no. 5, 3027–3076.
- [Hon00] K. Honda, *On the classification of tight contact structures. II.*, J. Differential Geom. **55** (2000), no. 1, 83–143.
- [JT12] A. Juhász and D. P. Thurston, *Naturality and mapping class groups in Heegaard Floer homology*, arXiv:1210.4996 (2012).
- [Juh06] A. Juhász, *Holomorphic discs and sutured manifolds*, Algebraic and Geometric Topology **6** (2006), 1429–1457.
- [Juh08] ———, *Floer homology and surface decompositions*, Geometry & Topology **12** (2008), no. 1, 299–350.
- [Juh09] ———, *Cobordisms of sutured manifolds and the functoriality of link floer homology*, to appear in Adv. Math., arXiv:0910.4382 (2009).
- [Juh10] A. Juhász, *The sutured Floer homology polytope*, Geom. Topol. **14** (2010), 1303–1354.
- [Juh14] ———, *Defining and classifying TQFTs via surgery*, arxiv:1408.0668 (2014).
- [Kho00] M. Khovanov, *A categorification of the Jones polynomial*, Duke Mathematical Journal **101** (2000), no. 3, 359–426.
- [Kho03] ———, *Patterns in knot cohomology, I*, Experimental mathematics **12** (2003), no. 3, 365–374.
- [KM11] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, Publications mathématiques de l’IHÉS **113** (2011), no. 1, 97–208.

- [Lut77] R. Lutz, *Structures de contact sur les fibrés principaux en cercles de dimension trois*, Ann. Inst. Fourier **27** (1977), no. 3, 1–15.
- [McC01] J. McCleary, *A user's guide to spectral sequences*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2001.
- [MO08] C. Manolescu and P. Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links*, Proceedings of Gökova Geometry-Topology Conference 2007 (2008), 60–81.
- [OS04] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Annals of Mathematics (2004), 1159–1245.
- [OSz04] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, Advances in Mathematics **186** (2004), no. 1, 58–116.
- [OSz05] ———, *On the Heegaard Floer homology of branched double-covers*, Advances in Mathematics **194** (2005), no. 1, 1–33.
- [OSz06] ———, *Holomorphic triangles and invariants for smooth four-manifolds*, Advances in Mathematics **202** (2006), no. 2, 326–400.
- [OSz08] ———, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebraic and Geometric Topology **8** (2008), no. 2, 615–692.
- [Ras03] J. A. Rasmussen, *Floer homology and knot complements*, Ph.D. Thesis, Harvard University, 2003.
- [Ras05] ———, *Knot polynomials and knot homologies*, Geometry and Topology of Manifolds **47** (2005), 261–280.
- [Sar15] S. Sarkar, *Moving basepoints and the induced automorphisms of link Floer homology*, Algebr. Geom. Topol. **15** (2015), 2479–2515.
- [Zem16a] I. Zemke, *Link cobordisms and functoriality in link Floer homology*, arXiv preprint arXiv:1610.05207 (2016).
- [Zem16b] ———, *Quasi-stabilization and basepoint moving maps in link Floer homology*, arxiv:1604.04316 (2016).
- [Zem17] ———, *Link cobordisms and absolute gradings on link Floer homology*, preprint (2017).

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